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HANDBOOK OF DATA REDUCTION METHODS

Darold W. Comstock, et al

White Sands Missile Range
White Sands, New Mexico

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DATA REDUCTION DIVISION

TECHNICAL REPORT

HANDBOOK OF DATA REDUCTION METHODS

13 August 1964

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WHITE SANDS MISSILE RANGE
NEW MEXICO, 88002

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Details of illustrations in this document may be better studied on microfiche

Darold W. Comstock

Darold W. Comstock (DRD-T)

Madeleine H. Wright

Madeleine H. Wright (DRD-T)

Virginia B. Tipton

Virginia B. Tipton (DRD-T)

ABSTRACT

This _Handbook of Data Reduction Methods_ contains summaries of reduction programs currently used by Data Reduction Division. Generally the program descriptions consist of the statement of the mathematical problem involved, the derivation of equations used and the computational procedure employed.

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TABLE OF CONTENTS

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A. GENERAL INTRODUCTION

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A HANDBOOK OF DATA REDUCTION METHODS

Introduction

This Handbook has been designed to provide in a single source summaries of the data reduction methods and programs currently used by the Data Reduction Division. Many of the techniques outlined in this Handbook are original developments of the authors. The basis for the other methods described were obtained from previously published technical reports and papers of individuals in the Data Reduction Division.

Particular care has been taken to insure that each of the sections of this Handbook can be viewed independently, as the "Program Requirements Document" for a particular reduction. In all cases possible the basic relationships among the physical principles involved have been used to show the detailed derivation of the equations programmed. In general, each section contains a description of the measurement process or mathematical problem involved, the derivation of the equations for the program, and a computational procedure to be followed. References are given for each section where available.

The authors wish to acknowledge especially the outstanding contribution of Mr. R.A. Montes, of DRD-T, without whose assistance this report would have been impossible.

B. POSITION DATA

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B. POSITION DATA

I Cinetheodolites

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WSMR CINETHEODOLITE REDUCTIONS

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INTRODUCTION:

Cinetheodolites are angle-measuring instruments used to determine the trajectories of rapidly moving aerial targets. Two or more theodolites placed at known distances from each other measure and record on film azimuth and elevation angles to the target. Since the distances between the theodolites (base lengths) and the angular measurements are known, the positions of the target in space can be computed.

The existing WSMR cinetheodolite system is composed mainly of two types of cameras, the standard Askania and the Contraves, which are located throughout the range to provide complete instrumentation coverage. The computational procedure is essentially the same for both types of theodolites.

In operation, the theodolites are continuously sighted on the target and photographs are taken of the target and two graduated circles (azimuth and elevation dials) at regular or irregular time intervals. To insure that these photographs are taken simultaneously at each measuring station, the shutters of the cameras are opened by electrical pulses from a central timing station.

Figure 1 illustrates the angles measured by two theodolites. The location of these theodolites depends on the type of terrain and on the flight or firing direction. The distance between any two stations should be a maximum of one-third and a minimum of one-fifth of the expected average distance between the camera and the target.

Figure 2 shows a film frame of a missile which was measured with Askania cinetheodolites. Figure 3 shows a film frame of an aircraft measured with Contraves theodolites.

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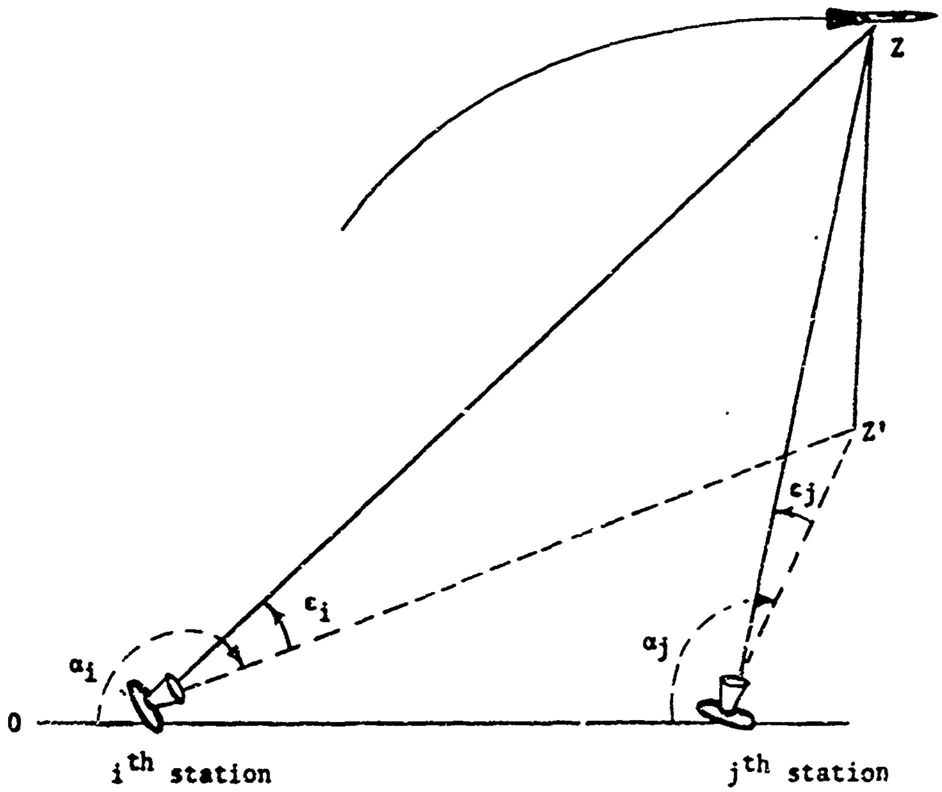


Fig. 1. Theodolite Geometry

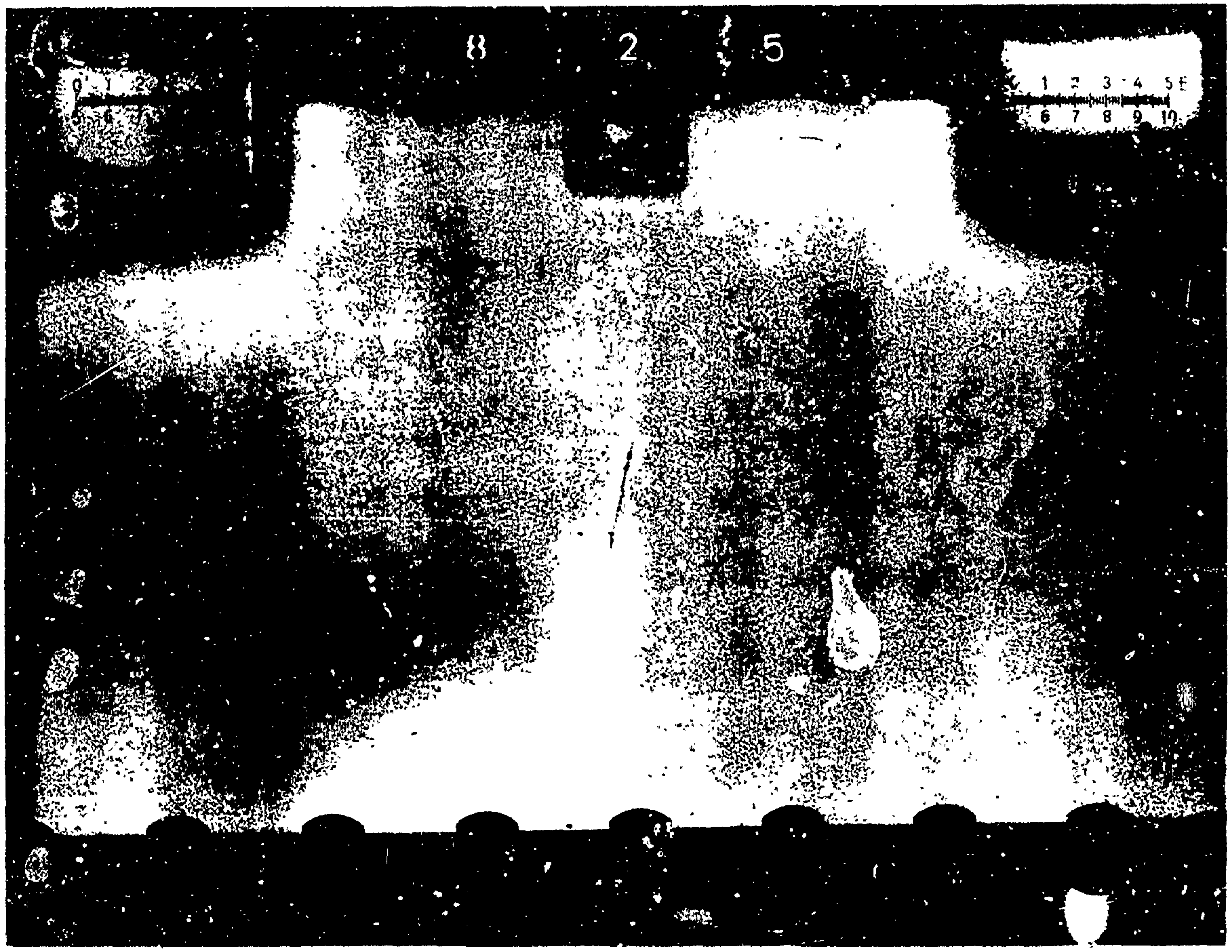


Fig. 2. Film Frame of Missile Measured with Askania Theodolites.

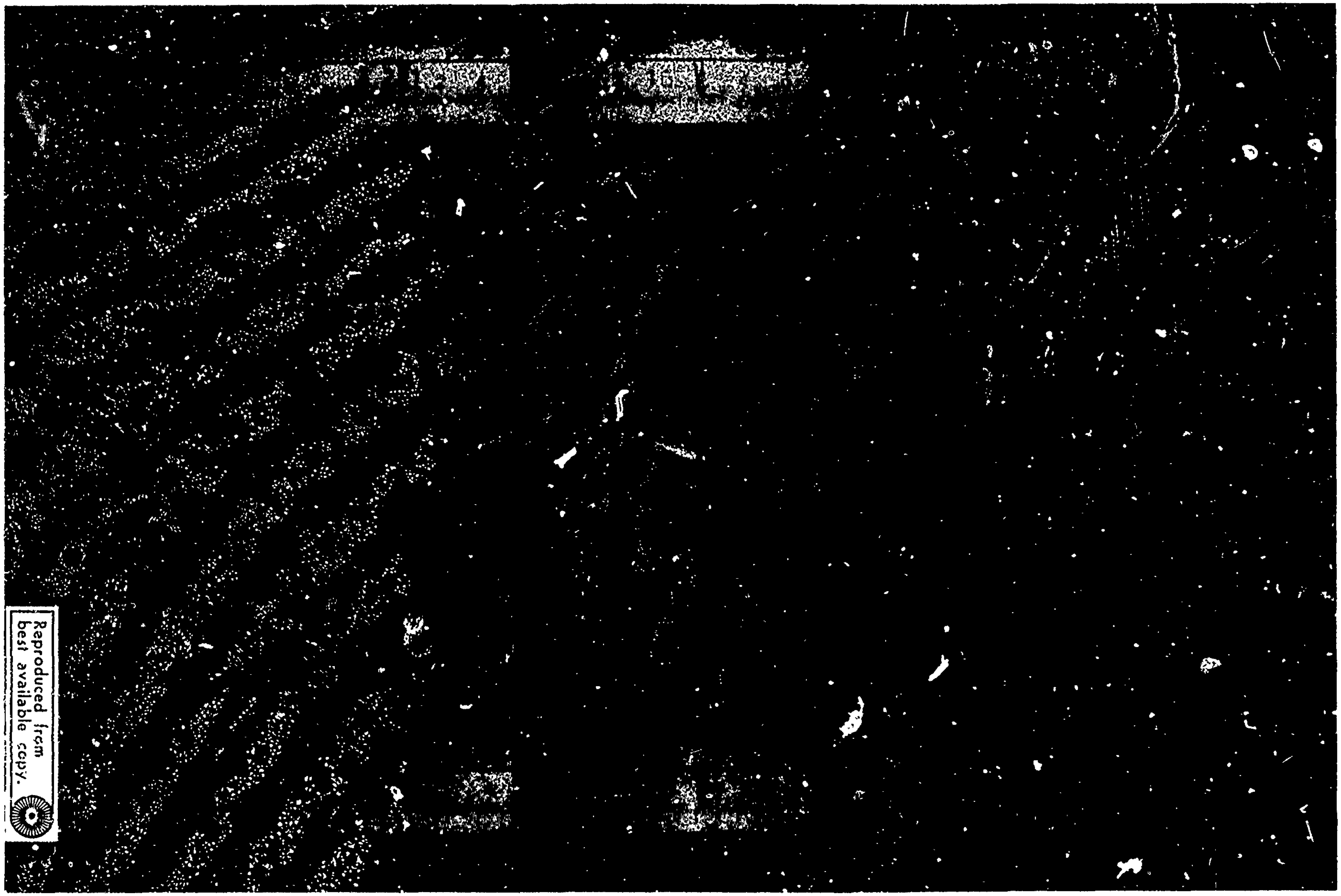


Fig. 3. Film Frame of Aircraft Measured with Contraves Theodolites

Description of the Askania Theodolite

The Askania theodolite is constructed about a rectangular axis system, illustrated in Figure 4, which consists of the vertical main axis, (19), the horizontal auxiliary axis (89), and the sighting axis (99) which is at right angles to the auxiliary axis. The three main parts of the cinetheodolite are:

1. The rigid base, Fig. 5(14).

2. The U-shaped camera trunnion, Fig. 5(46), which turns about the vertical main axis.

3. The camera, which can be tilted through the horizontal auxiliary axis, with objective lens, Fig. 4(105) and sighting telescopes, Fig. 4(70,71).

The theodolite is secured to the base plate by three foot-screws and holding clamps. The camera trunnion, which includes two levels at right angles to each other, Fig. 5(44), is leveled with the horizontal axis by means of these foot-screws.

The trunnion also contains the operators' eyepieces for the tracking telescopes, Fig. 4(49), and the necessary optics for projection of the angle data from the graduated circles onto the film.

Sighting the Theodolite

On opposite sides of the camera carrier, hand wheels, Fig. 5(37,38) are provided so the camera can be turned by worm gears, Fig. 6(58), and worm wheels around the vertical main axis, or tilted about the horizontal auxiliary axis. The hand wheels have a two-speed transmission, coarse (three degrees per revolution) and fine (one degree per revolution). The switching from one to the other is done by pushing or pulling the hand wheels in the direction of the axis. To set the camera quickly, the worms can be released so the camera is freely sighted toward the target. The lateral worm drive is released by turning the knob, Fig. 6(28), counter-clockwise, whereas the vertical drive is released by turning the lever, Fig. 6(59), to the right.

Graduated Circles

The theodolite is provided with two glass circles, which center with the main axis or the auxiliary axis, respectively. The azimuth circle, Fig. 4(31), is graduated every .5° and numbered clockwise from 0° to 360°. It may be manually zeroed by the field operator. The elevation circle, Fig. 4(50), is graduated from 0° to 180° and calibrated for an additional graduation of approximately 10° in both directions. The vertical motion gears disconnect at a depression angle of 6°, which limits elevation traverse.

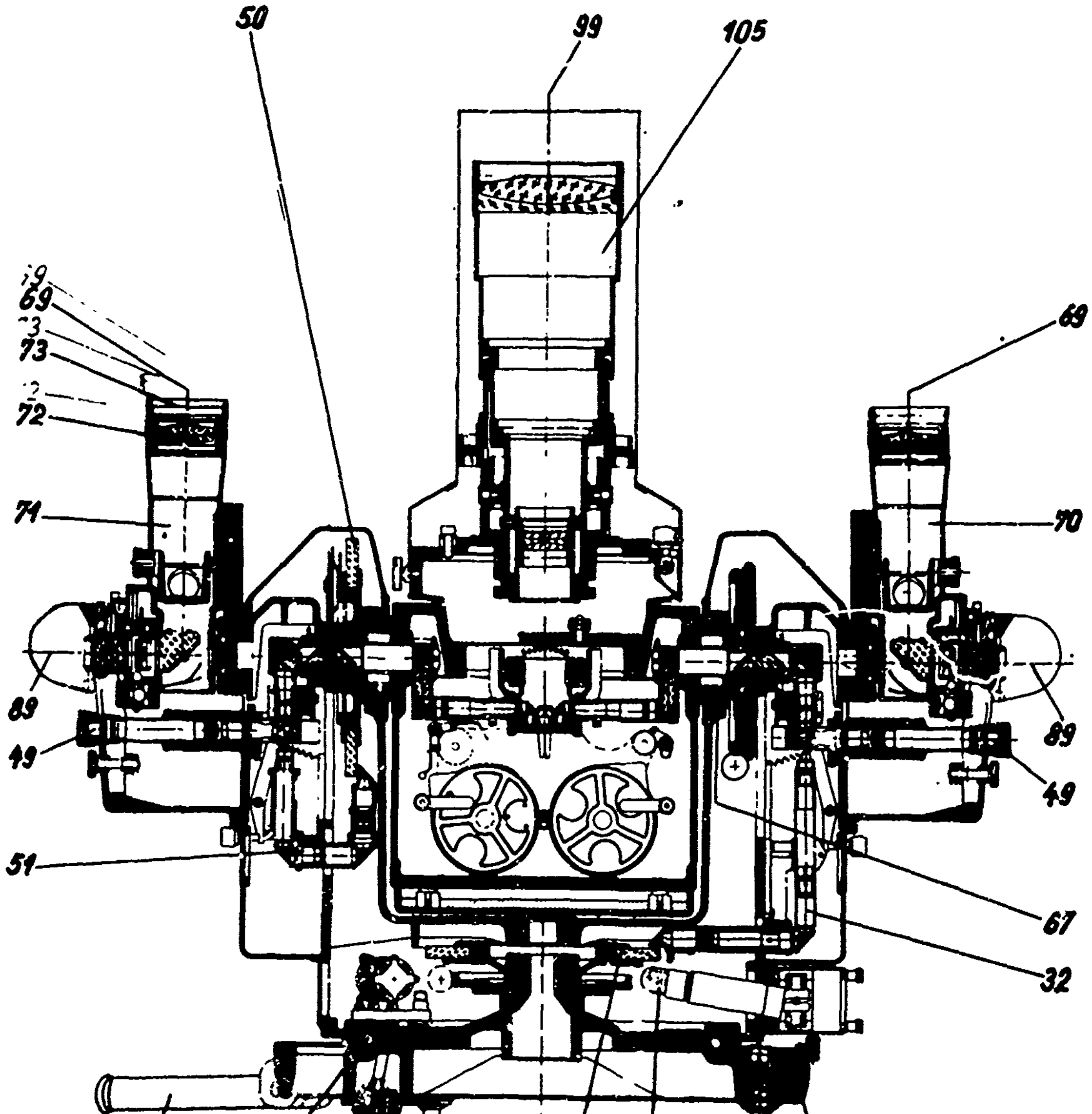


Fig. 4. Cinetheodolite Kth 53, Cross Section.

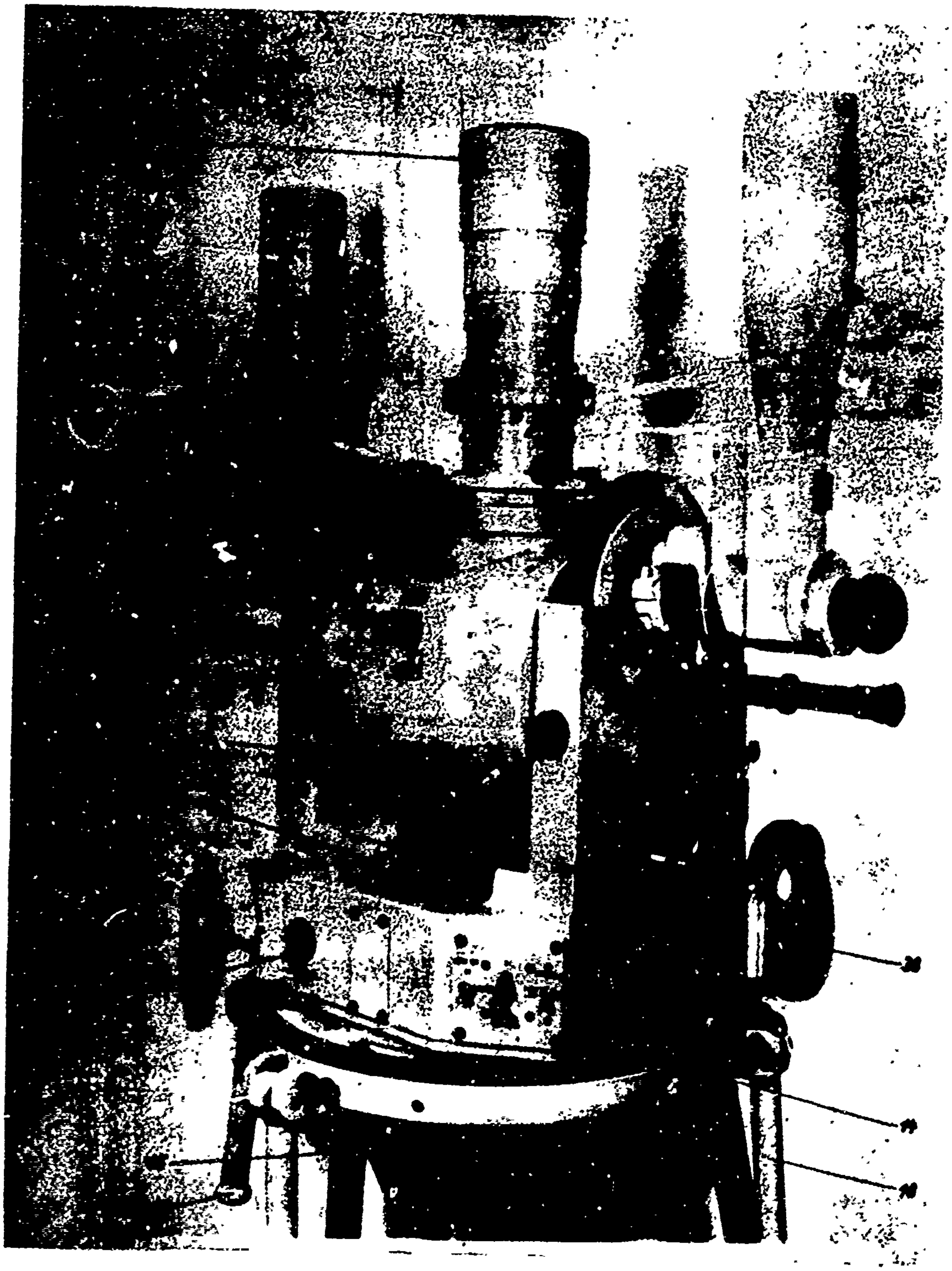


Fig. 5. U-Shaped Camera Trunnion

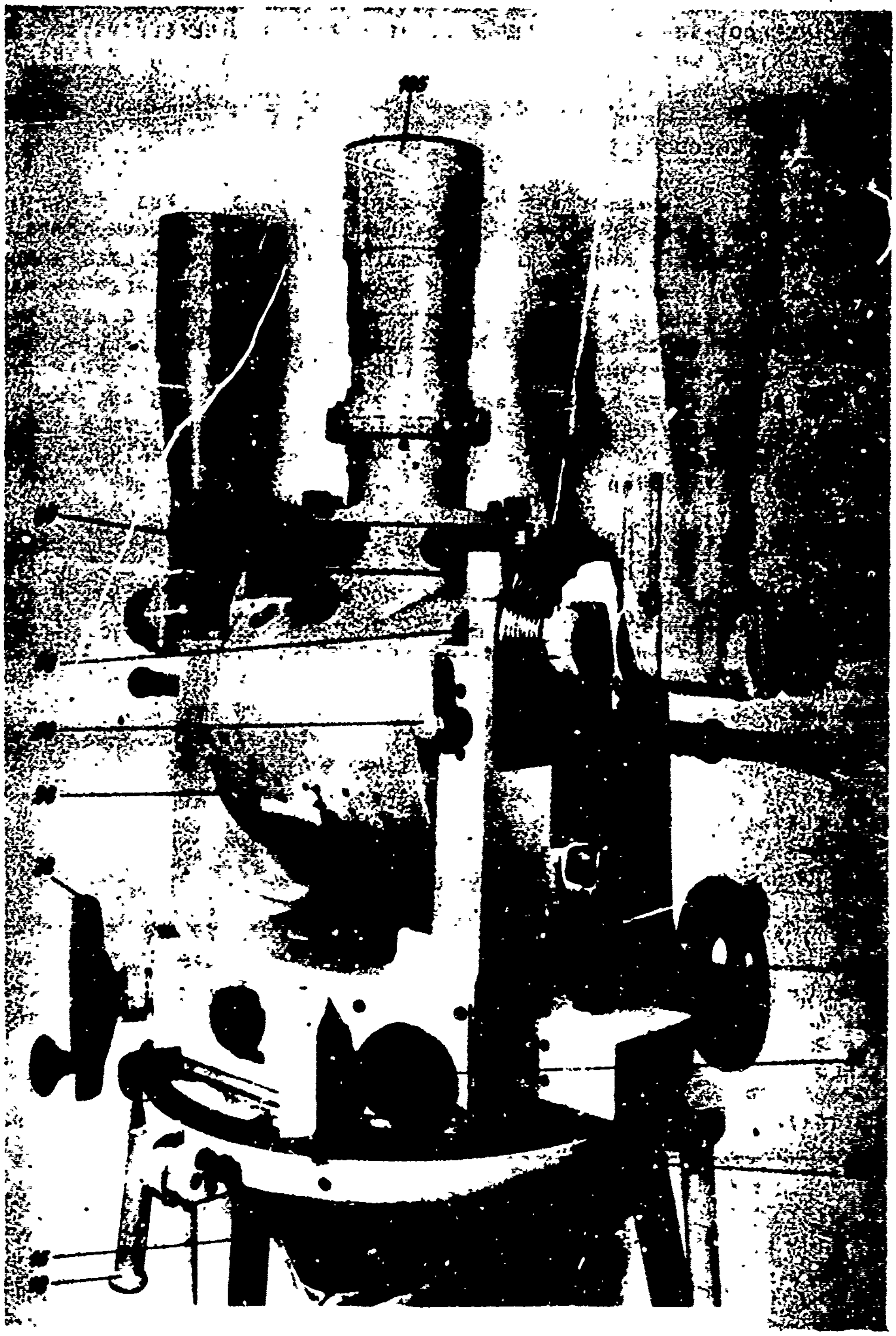
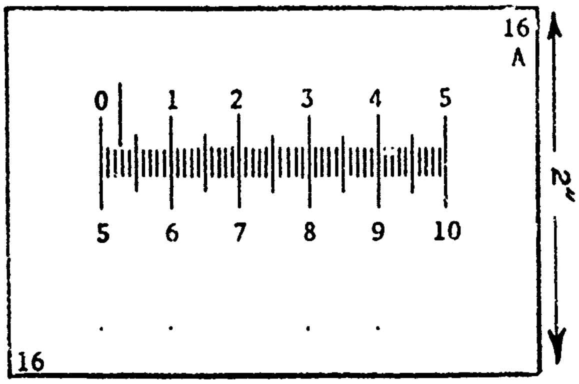


Fig. 6. Cinetheodolite, Worm Gear for Elevation

Reading Microscope and Photo Recordings

Optical systems, Figs. 4(32,51), located inside the camera carrier, in the hollow auxiliary axis, and in the camera itself, are used to read the glass circles and film recordings. The etched scale divisions on the glass circles are first projected on a scale plate and then transmitted by various prisms to the aperture plate. Microscopes, Fig. 4(49), switched into the optical ray path by a push button, are used in reading the graduated circles which are calibrated to 0.01° so that 0.001° can be estimated.



This dial reads 16.528. If the number 16 were in the upper left hand corner the dial would read 16.028.

Fig. 7. Askania Dial Reading (Style A)

New Askania cameras show whole degree values. In addition, each degree is coded at the bottom of the dial.

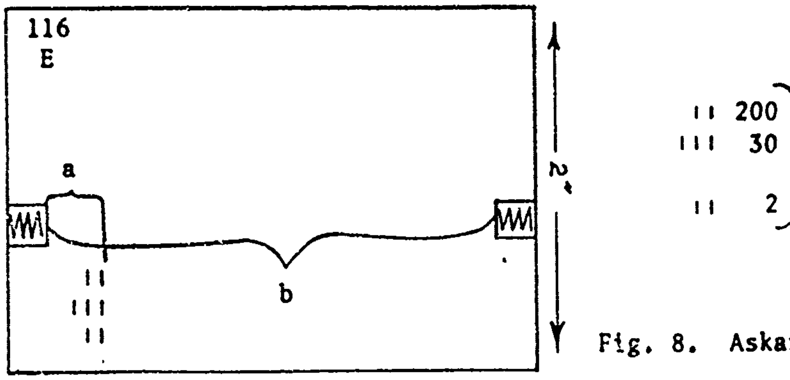


Fig. 8. Askania Dial Reading (Style B)

The fraction of the degree would be 0.125°. Since the distance, b, represents one-half degree $\frac{a}{b}$ represents a decimal portion of that half degree. Therefore, the angle is 116.125°. Had the whole number 116 been in the lower half of the dial, the actual angle would be 116.5° plus the fraction $\frac{a}{b}$ or 116.625°.

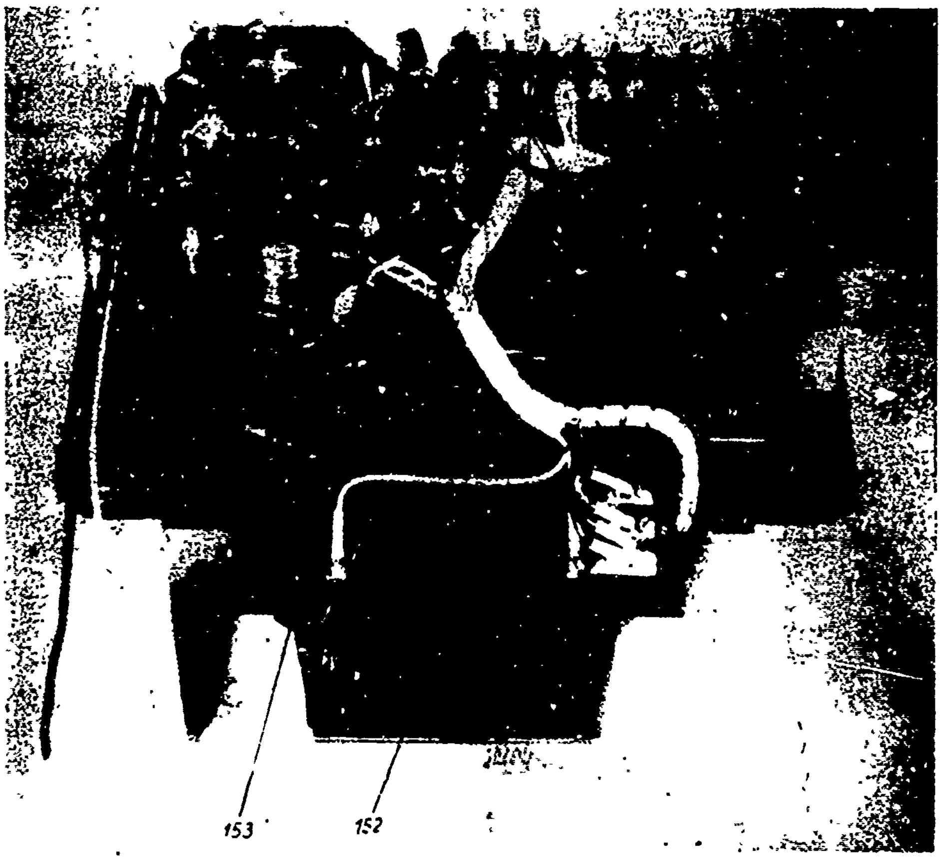


Fig. 9. Askania Camera, Counter

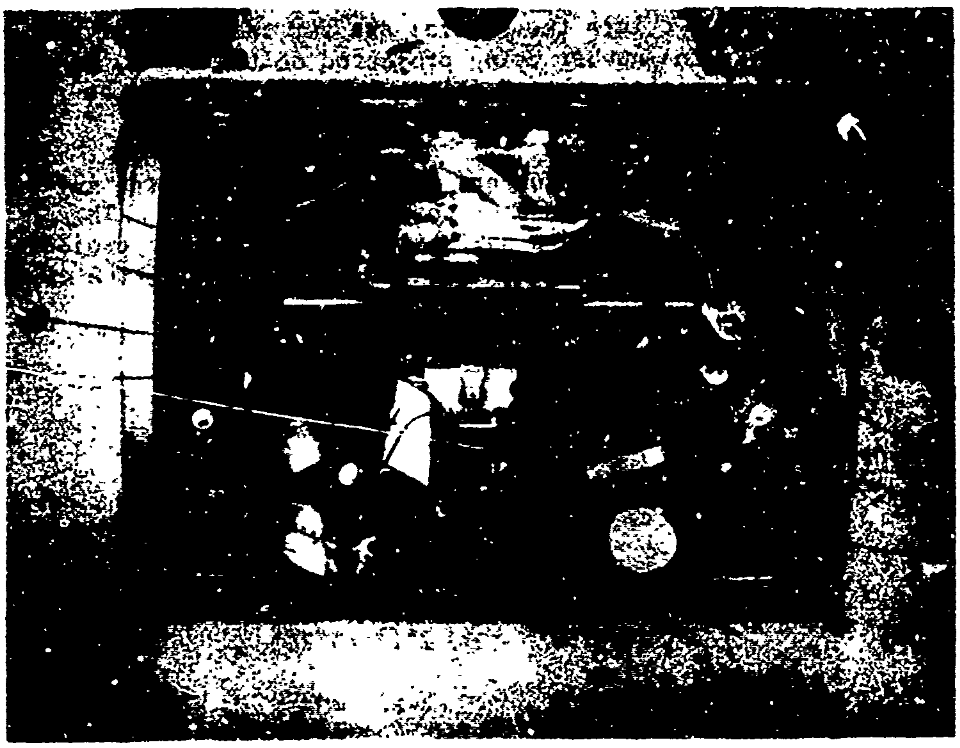


Fig. 10. Askania Camera, Magazine side

Camera

The camera consists of one object lens, Fig. 4(105), with aperture plate, camera housing, shutter, the camera mechanism, motor, and film transport gears.

Sighting Telescopes

Each side of the camera has a sighting telescope, Fig. 4(70,71). These are inserted in guiding grooves and fastened by clamp screws, Fig. 5(87). The operators position the camera on target by turning the hand wheels until the target is centered in the field of view of the telescopes.

Frame Counters

Two counters are built into the camera: one for the consecutive numbering of the frame, the number being projected on the frame, and the other indicating the amount of exposed film. Both counters can be read from outside the camera when it is open.

Before operations the frame counters of all the theodolites may be set to zero to facilitate comparisons of corresponding frames. The number shown in the window is always one unit smaller than the projected number on the film. If the frame series is to begin with 000, the counter should be set to 999.

Technical Data for Askania

Objective Lenses:

Focus of	30 cm	60 cm	100 cm
Aperture	1:4.5	1:4.5	1:6.3
Field of view	4°36'x6°56'	2°18'x3°28'	1°23'x2°04'
Frame size	24x36 mm	24x36 mm	24x36 mm
Diaphram	1:42	1:42	1:8 and 1:11

Sighting Telescopes:

Magnification	10x	12x	20x
Field of view	6°8'	5°25'	3°13'
Free aperture	60 mm	60 mm	80 mm

Graduated Circles:

Graduated from one-hundredth degree. Each degree and each one-half degree is numbered.

Azimuth	0-360°
Elevation	0-180° with additional graduation of 10° in both directions.

Reading Microscopes:

Fine Reading through 100 division scale to 1/1000 degree.

Magnification	35x
Venetian Blind Shutter	Exposure about 1/150 second.
Maximum Frame speed	5 frames per second.
Magazine capacity	To 165 feet standard film.

Description of Contraves Theodolite

The Contraves photo-theodolite, Fig. 11, is an instrument which when directed at stationary and moving targets records their positions in angle values at accurately determined time intervals. At each of these intervals the target is photographed and on each of these photographs the position is indicated with respect to cross hairs, azimuth, elevation and time. A theodolite measuring system is comprised of at least two theodolites. To correct the measured data for instrument errors, each theodolite is provided with an array of fixed target boards. The positions of the target boards are determined in the WSCS coordinate system with high accuracy by the fundamental survey of the installation.

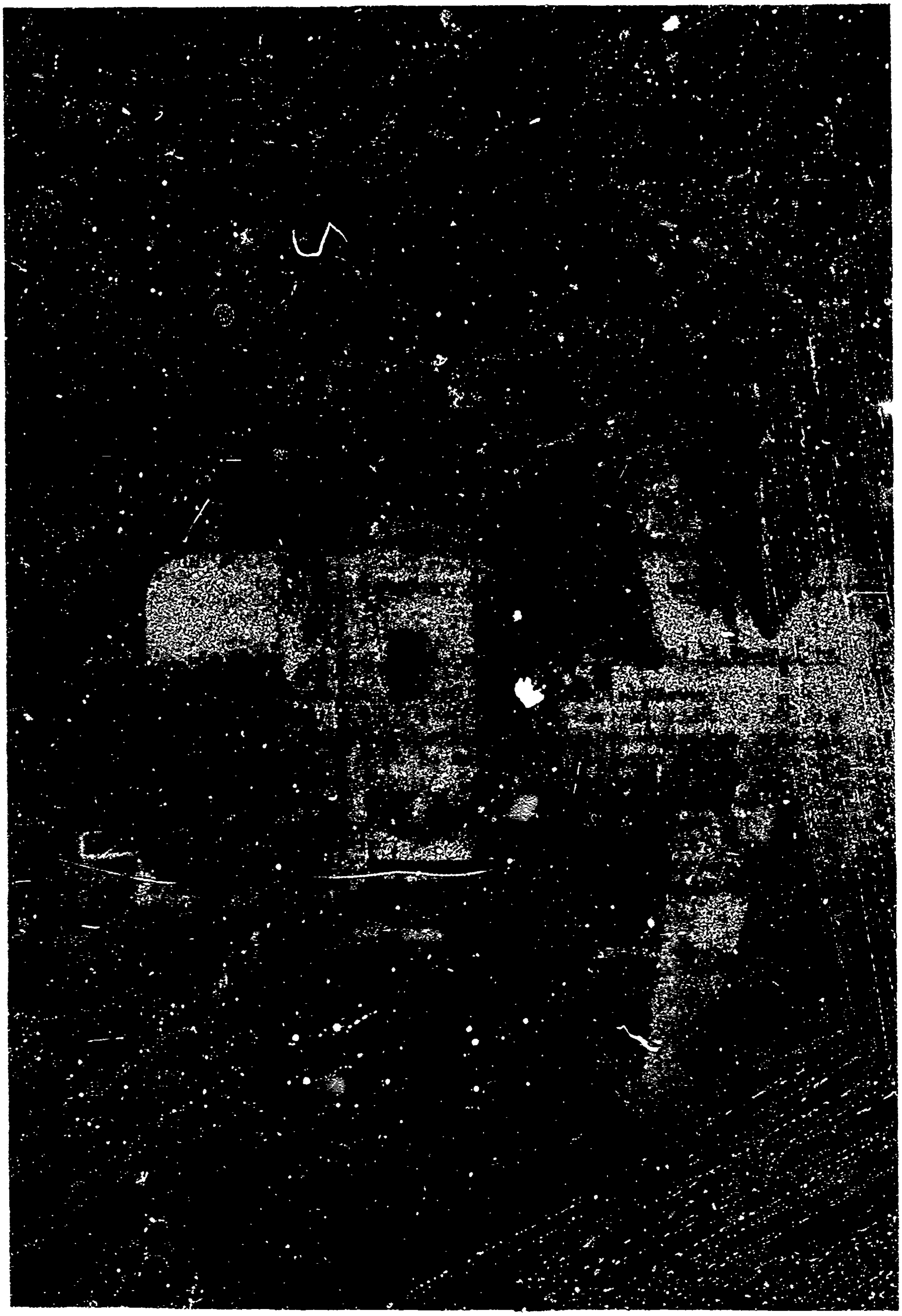


Fig. 11. Contraves Phototheodolite

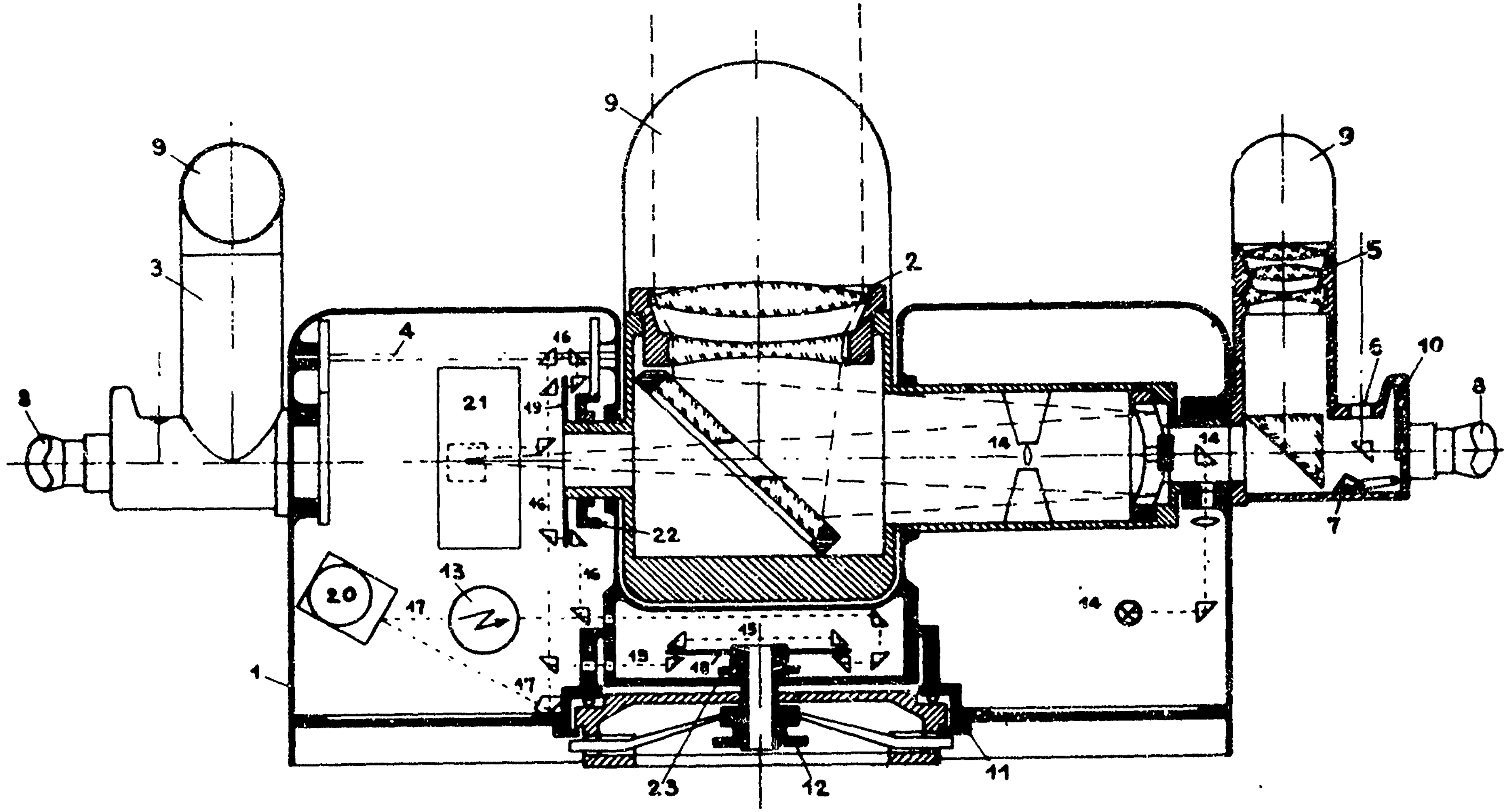


Fig. 12. Contraves Phototheodolite, Cross Section

Description of Figure 12

1. Optics housing
2. Main telescope
3. Tracking telescope
4. Elevation drive shaft
5. Tracking telescope objective-12, 4 power
6. Tracking telescope objective-2 power
7. Prism for magnification selection
8. Eye piece
9. Sun shield
10. Color filter turret
11. Azimuth bearing
12. Image plane
13. Flash lamp
14. Light path for cross hair projection
15. Light path for azimuth scale projection
16. Light path for elevation scale projection
17. Light path for frame count projection
18. Glass disc with azimuth scale engraving
19. Glass disc with elevation scale engraving
20. Frame counter
21. Synchronized camera-35 mm
22. Spur gear for elevation drive
23. Spur gear for azimuth scale setting
24. Cross hair illumination bulb
25. Reading microscope

Optical System

The optical system of the theodolite, Fig. 12, is comprised of three sections: the angle measuring equipment, the photographic optical equipment and the sighting equipment. The angle measuring equipment consists of two double graduated circles which are used to reproduce on film the angular position of the main telescope. The circle graduations are projected into the camera by an optical system using prisms for deflection and a flash lamp as the source of illumination. The photographic optical equipment consists of the main telescope, an iris, filters, focusing mechanism, cross hairs and camera. The main telescope is a folded optical system with a focal length of 1.5 meters and an aperture of f 1:8. The sighting equipment consists of two tracking telescopes, one for elevation and one for azimuth. They are provided with a change-over prism for changing magnification from 2X for searching to 12.4X for tracking.

The camera takes pictures through the main telescope at the rate of either five, ten, twenty, or thirty frames per second. Standard 35mm motion picture film is used with a register pin movement. The picture, Fig. 3., is 0.8° x 0.5° located in the upper three quarters of a standard 35mm single frame picture. Azimuth, elevation and frame number dials are simultaneously photographed across the bottom of the 35mm frame. Two perpendicular cross hairs are located in the center of the picture on the optical axis of the main telescope. These are used as reference marks to determine tracking corrections between the image of the target and the optical axis of the telescope as indicated by the cross hairs and the azimuth and elevation dials.

Time Recording System

The time recording system consists of a flash lamp combined with a frame counter. Very accurate timing pulses are used to trigger the flash. The same pulses are used to operate the electro-mechanical frame counter, serving actually as a counter of the received timing pulses. The flash illuminated frame counter scale is projected via its own set of deflecting prisms and lenses into the camera. A microscope enables the operator to read the two double-circles and the frame counter.

One Man Tracking

The tracking system has been developed to provide for one man operation. The conventional two man handwheel operation is replaced with a joy-stick. Through careful proportioning of the controlled position-velocity-acceleration drive, good control response characteristics are achieved.

Reading Microscope and Photo Recordings

Readings of the azimuth and elevation are presented on the film from graduated circles. The scale is read in two steps: First, the main scale is read with respect to the pointer (coarse index) to the nearest 0.1 degree which is represented by the numerals and larger graduations of the main scale, (Fig. 13). Second, the fine index position is read to provide 0.01 degree directly and 0.001 degree by interpolation. This is done by starting the fine index reading under a large main scale graduation which is numbered, and reading with respect to the main scale graduations. Although it appears the scale can be read at any of five positions, the recommended position is the one in the center of the field to minimize, to a negligible amount, any distortional effects of the data optical system.

The small graduations of the main scale represent 0.02 degrees when reading only the main scale with reference to the fixed pointer, but these same small graduations represent 0.01 degree when reading with respect to the oppositely moving fine index scale.

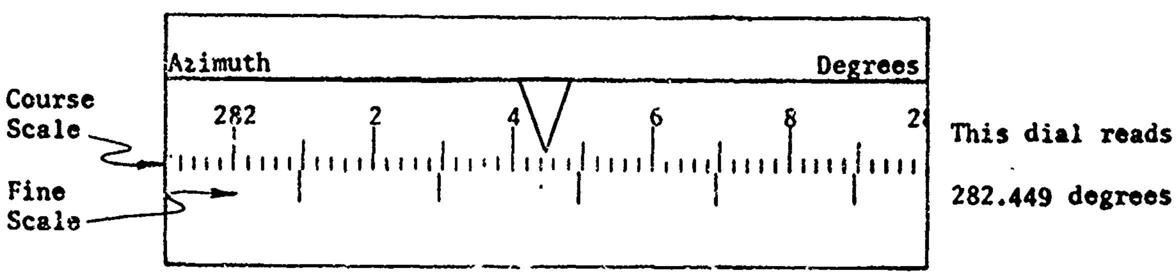


Fig. 13. Contraves Dial Reading

Reticle Pattern on the Film

The reticle, two orthogonal lines, rotates in elevation. The two short perpendicular marks on the same reticle line are calibration marks which are a reference for film shrinkage computation. It is possible to read tracking corrections using the reticle but with existing equipment at WSMR it is too time consuming. If the reticle is used to measure the tracking corrections, the reticle with the two short perpendicular marks is the line along which the azimuth tracking error is read, additive error to the left of the elevation, subtractive to the right. The third short perpendicular mark indicates the line along which the positive elevation tracking error is read.

Technical Data for Contraves

Objective Lenses:

Focal Length	1500 mm ±5% (60 in)
Aperture	190 mm (7.5 in)
Relative aperture	1:8
Iris Diaphragm	1:8 to 1:22
Frame size	24 x 15 mm, 0.8° x 0.5°

Sighting Telescopes:

Magnification	2 and 12.4 power
	4 and 20 power optional
Field of View	25.5° and 5.5°
	12.5° and 2.5° with option

Operating Range

Elevation	-5° to +185°
Azimuth	0 to 360°

Camera Frame Rate 5, 10, 20 and 30 frames per second

FILM READING EQUIPMENT

TELEREADEX - TYPE 29-A (FRONT PROJECTION TELEREADER)

The Telereadex (see Figures 14 and 15) is a front-projection film reader incorporating positive pin registration and automatic frame advance. The image is projected by a 750-watt lamp through a lens directed upward where it is reflected by a silver surfaced mirror onto a flat white surface to give maximum resolution. "X" and "Y" boresight measurements are made using movable cross hairs which traverse the entire surface. Associated recording equipment includes the Telecordex Linear Calibrator, either the IBM 517 or 523 Summary Punch, and an automatic typewriter.

The machine can take either standard double frame or high speed single frame Askania or Contraves Film, or 16mm and 35mm radar boresight film. The majority of radar boresight and high speed Askania films use the pin registration; the standard Askania does not. There are 16mm, 35mm and 70mm mechanisms available. The film may be advanced 1 to 12 single frames or 1 to 6 Askania frames.

The lens system of the Telereadex consists of 5, 10 and 20 power magnifications with 40 power available if needed.

The accuracy of the machine varies with the steadiness of the initial tracking, pin registration, and film quality. Accurate measurements can be made at speeds of 7.2 seconds per frame on standard Askania; 2.2 seconds per frame on high speed Askania and pin registered 16mm radar boresight. Pin registration is accurate to .003 inch. Least count is ±110 microns at the measuring surface.

DIAL READERS

The Coleman Dial Reader (see Figs. 16 and 17) is an electro-mechanical apparatus for reading dials on either standard or high speed Askania 35mm film. The projector lens, film, and film movement mechanism are all attached to a movable carriage which may be shifted by means of a side-mounted lever, or rotated by a worm gear, to position the image on the frosted glass surface for viewing.

The film may be advanced manually or automatically by a selector switch on the front panel. At the discretion of the operator it may be advanced one through twelve frames, or continuously. Film may be run forward or backward with the frame count adding or subtracting, depending on the direction of film travel.

Two vertical wires are positioned just behind the viewing plate for measurement of either azimuth or elevation, with separate runs for each measurement. The two wires are so coupled that one wire is used to set zero on the viewing surface, and the other wire, which is coupled to a digitizing unit for the memory relays, is moved to make the measurement.

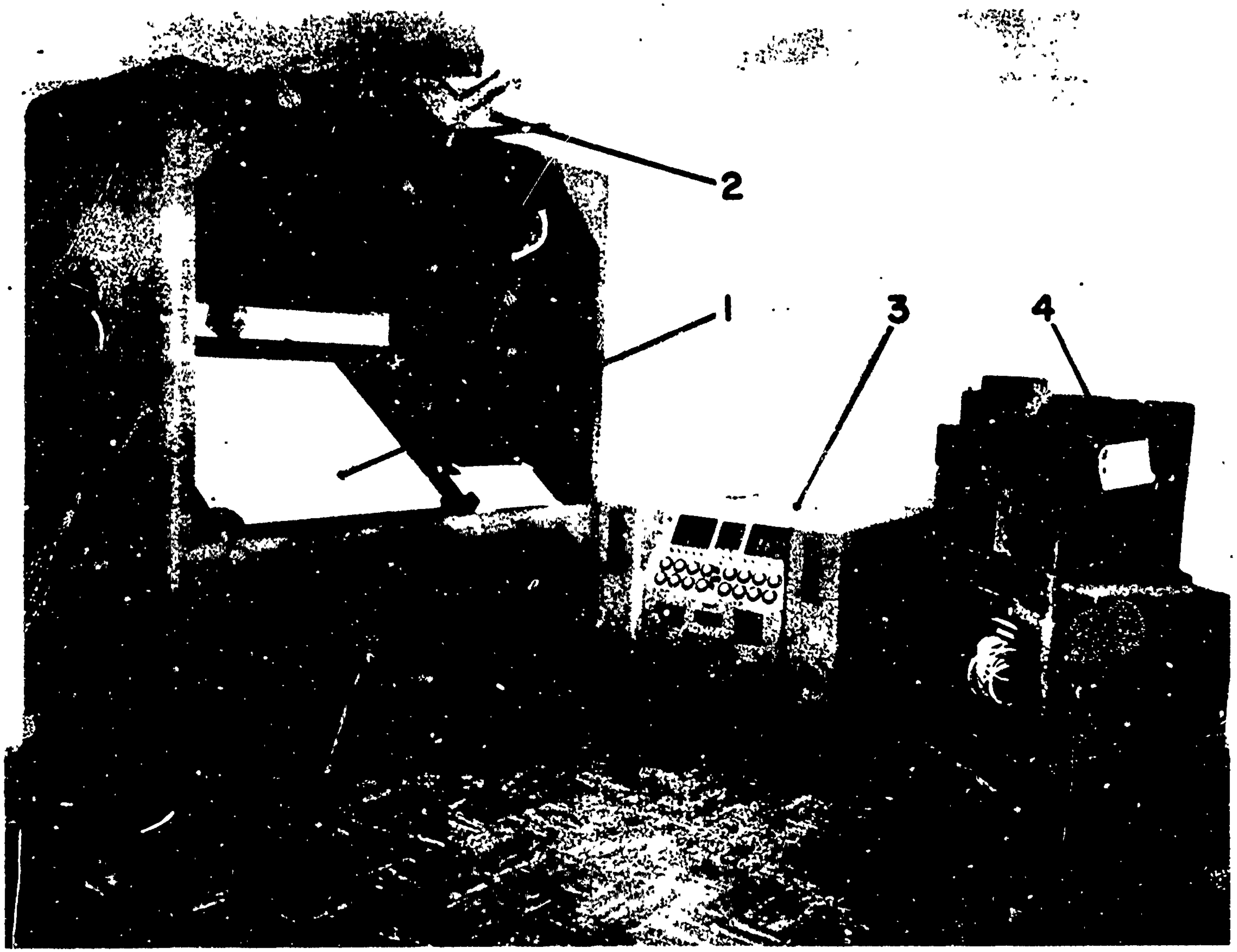


Fig. 14. FRONT PROJECTOR AND AUXILIARY EQUIPMENT

1. Telereader, Type 29A (Front Projector)
2. Reflecting Mirror
3. Telecordex
4. IBM Card Punch

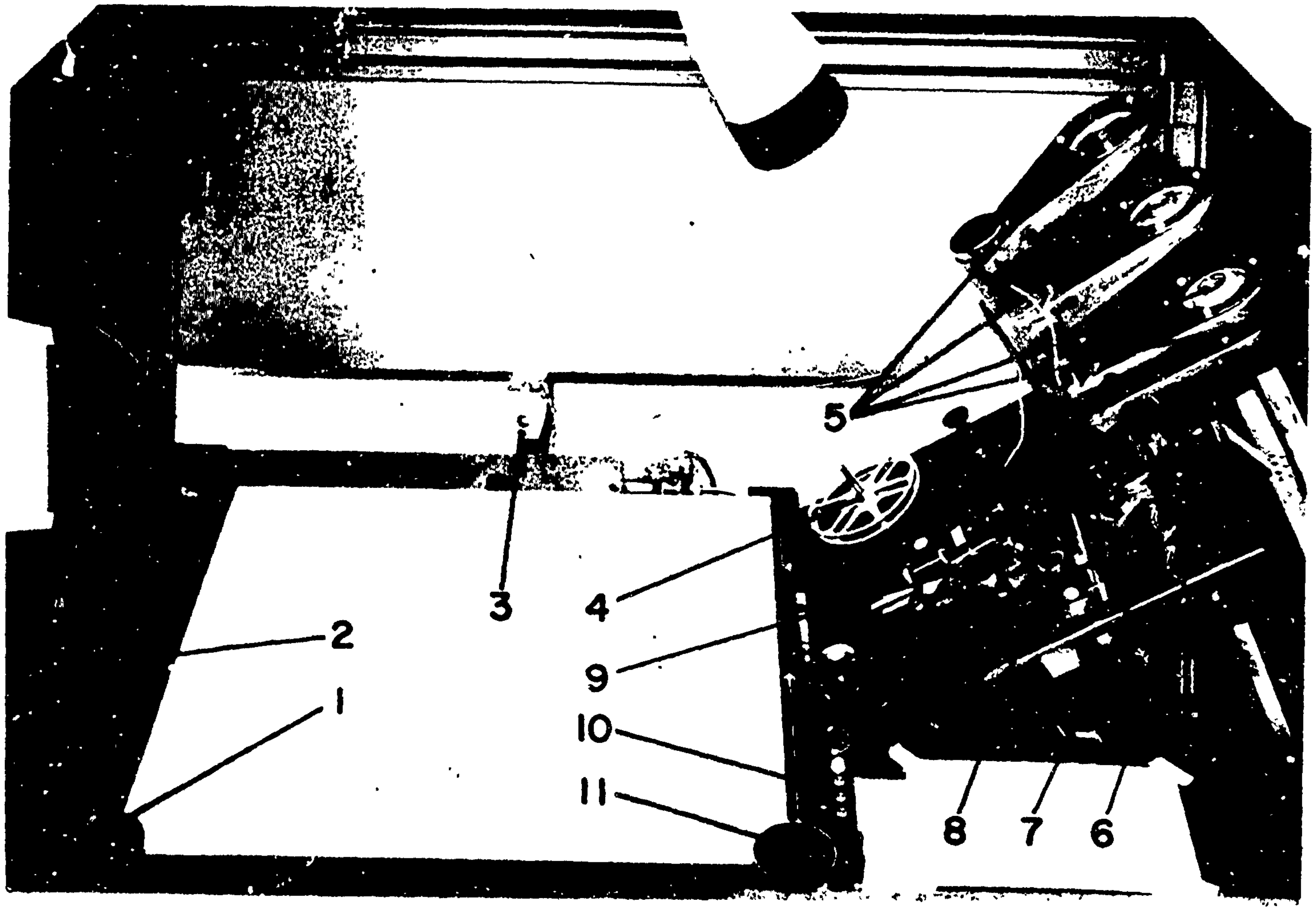


Fig. 15. CONSOLE FRONT PROJECTOR

1. Horizontal Cross Hair Control
2. Frame Advance Selector Switch
3. Line Voltage Indicator Light
4. Film Spool
5. Magnifier Lenses
6. Vertical Film Adjustment
7. Horizontal Film Adjustment
8. Rotation Film Adjustment
9. Film Advance Mechanism (16mm)
10. Counter Clear Switches (X, Y, X&Y)
11. Vertical Cross Hair Control

Fig. 16. CONSOLE, COLEMAN DIAL READER

1. One-Half Degree Count
2. Hairline Adjustment
3. Frame Adjustment
4. Light Density Knob
5. Whole Degree Keyboard

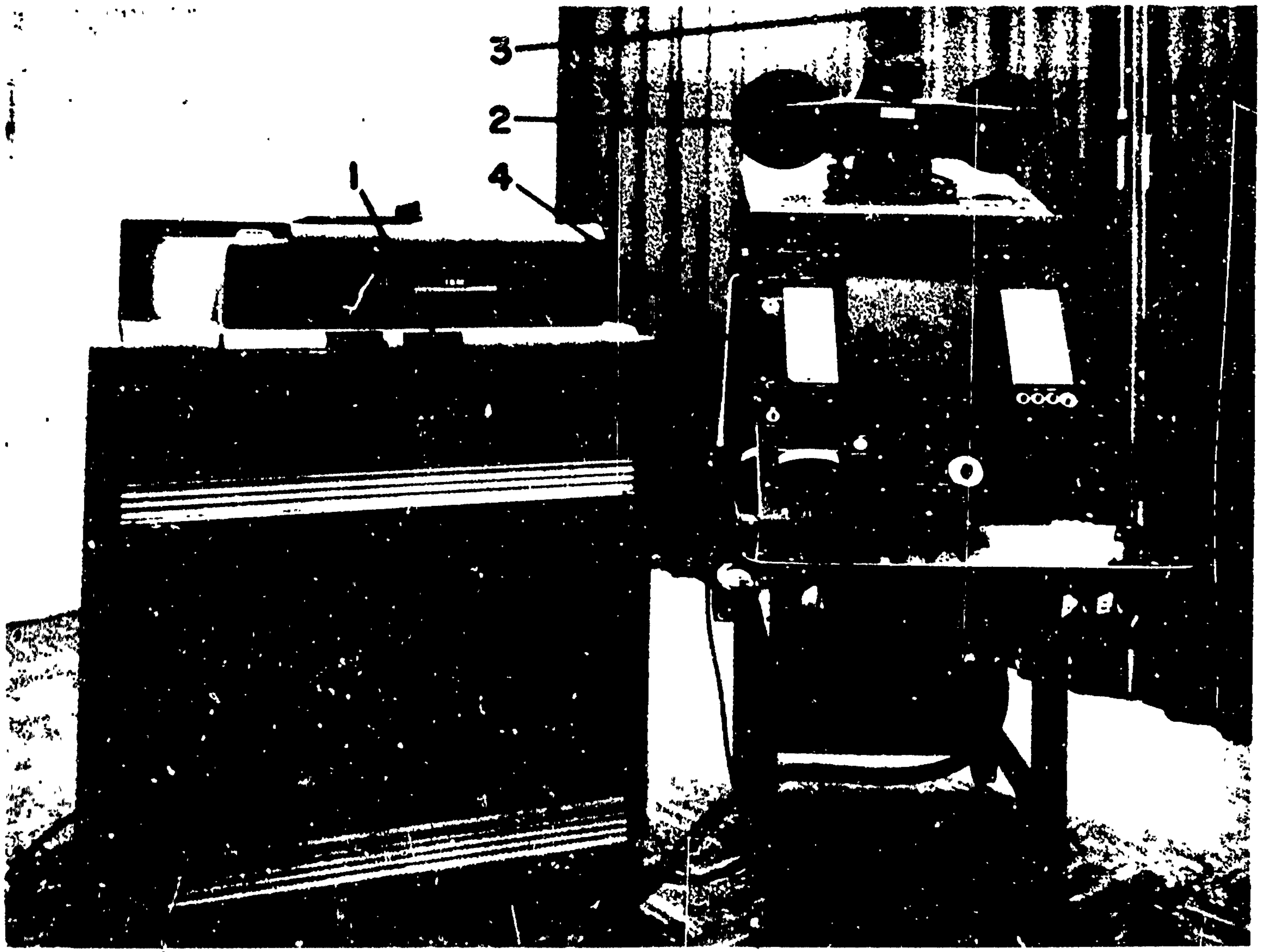


Fig. 17. COLEMAN DIAL VIEWER WITH PUNCH

1. IBM Card Punch
2. Film Transport Housing
3. Projection Light Housing
4. Console

Readings may be made at the rate of approximately three hundred per hour with high speed Askania, or two hundred per hour with standard Askania with the digitizing vernier of one part in one thousand. The whole degree readings are punched in a keyboard and the +180° and +.5° indicator switches set. These data are read out into an IBM 517 or 523 Summary Punch furnishing azimuth or elevation readings, frame count, the whole degrees, and switch settings on punched cards.

The Coleman dial reader has been modified to read Contraves dials.

The Parabam and Corvus dial readers are essentially the same as the Coleman. The Parabam dial reader utilizes 35mm or 70mm film. Contraves film can not be read on the Parabam. The Corvus dial reader is adaptable to either Contraves or Askania film.

ASKANIA VIEWER

The Askania Viewer is a portable manually operated device for viewing and measuring boresight data recorded on 35mm film. With the aid of two cross hairs and the calibrated degrees and minutes marks around the upper edge of the plate, it is possible to read angular measurements. The cross hairs are marked in degrees and minutes, both vertically and horizontally, so that it is possible to read tracking corrections directly from Askania film. The azimuth and elevation dials may also be read with the Askania viewer, making it possible to obtain all the required information to compute space positions from the one machine.

The viewer requires only 12 by 18 inches of space; however, it is usually mounted between two large hand cranked reels on an 18 by 42 inch table to make transportation of the film easier. Operated in this manner, the machine has a capacity of 800 feet of 35mm film. The film transport on the viewer itself is limited to a total capacity of 100 feet of film.

The film is wound from one to the other of the two hand-operated reels, passing over a diffused light and being viewed through the machine during this process. The diffused light is furnished by a 25-watt frosted bulb located in a metal cylinder in front of the machine. The light is reflected from a diffusion plate through the film. Two operators, one reading and one recording, can read and record 40 to 50 frames of Askania film per hour. This includes the reading of azimuth dials, elevation dials, X and Y displacement readings, allows time to change film rolls, and mark frames to be read. The viewer is calibrated to read directly from film taken by 60 cm optics.

ASSESSING AND READING FILM

The purpose of reading film is to determine the location of the target or missile with respect to the center of the frame. The Telereadex (Front Projector) is the machine primarily used for these measurements. When reading Contraves film the head of the reader must be rotated 90 degrees so that the film X and Y axes coincide with the reading machines' X and Y cross hairs.

Assessing Film

The following information appears on the films:

1. Timing
2. Direct and Indirect orientation targets (6-8 each)
3. Missile images
4. Azimuth and elevation dials

The film is assessed for the following factors:

1. Quality, density and contrast of film in general
2. Definition of targets and missile images
3. Definition of fiducial points
4. Definition of azimuth and elevation dials
5. Definition of frame counters
6. Definition of time code
7. Consistency of missile remaining in each frame through the complete run.

The following additional information is noted during assessment as an aid to the reader:

1. Frame number of zero time for each camera
2. Number of readable frames
3. The numbers of the first and last frames containing readable images.

READING ORIENTATION TARGETS

There is a maximum of eight direct and eight indirect targets. Six to twelve frames are taken of each target (direct and indirect). The machine operator runs the film to the first target and selects the best target image for reading. The machine is then zeroed by moving the cross hairs until they coincide with the fiducial marks on the Askania or the intersection of the orthogonal lines on the Contraves film. The operator then clears the Telereadex and moves the cross hairs to the center of the target image. The displacement (x, y) of the target image from the center of the frame is punched on cards by the IBM card punch. The operator then selects the best image of the second target, zeroes the machine, and reads the displacement of that image. This is done for each of the targets.

After all the targets (direct and indirect) have been read, the film is run to the first readable missile image marked by assessments. The above procedure is followed for reading missile images. Each frame is zeroed, read, and punched on cards as requested by the projects.

DIAL READINGS FOR 35MM FILM

The boresight readings give the image location (target or missile) with respect to the center of the frame. The dial readings give the angular position of the optical axis, which goes through the center of the frame. This brings the internal system of the camera into alignment with the external system of the range. The film should be run to the first target image recorded, the azimuth dial read, and the information punched on the card. The azimuth dial is read for every frame which was measured in the orientation and boresight corrections. After the azimuth dial corresponding to the last missile image is read and recorded, the operator returns to the first target and repeats this procedure, reading the elevation dials.

To summarize, three sets of cards are punched: One set with the orientation targets and missile displacements, one set with the azimuth dial readings, and the third set with the elevation dial readings. These three sets of cards are combined and fed to the IBM 7094 for computing corrected angles and trajectory data.

TIMING ON ASKANIA FILM

A binary coded decimal timing is used on the Askania film. While the film moves from left to right, the code is read from right to left. The code is always at the bottom of the film and recycles every ten minutes. Zero time is indicated by three reference marks with no coding indicated. There are also reference marks between decimal digits. The code from 9 minutes, 51 seconds through ten minutes as it is seen on the film is shown in Fig. 18.

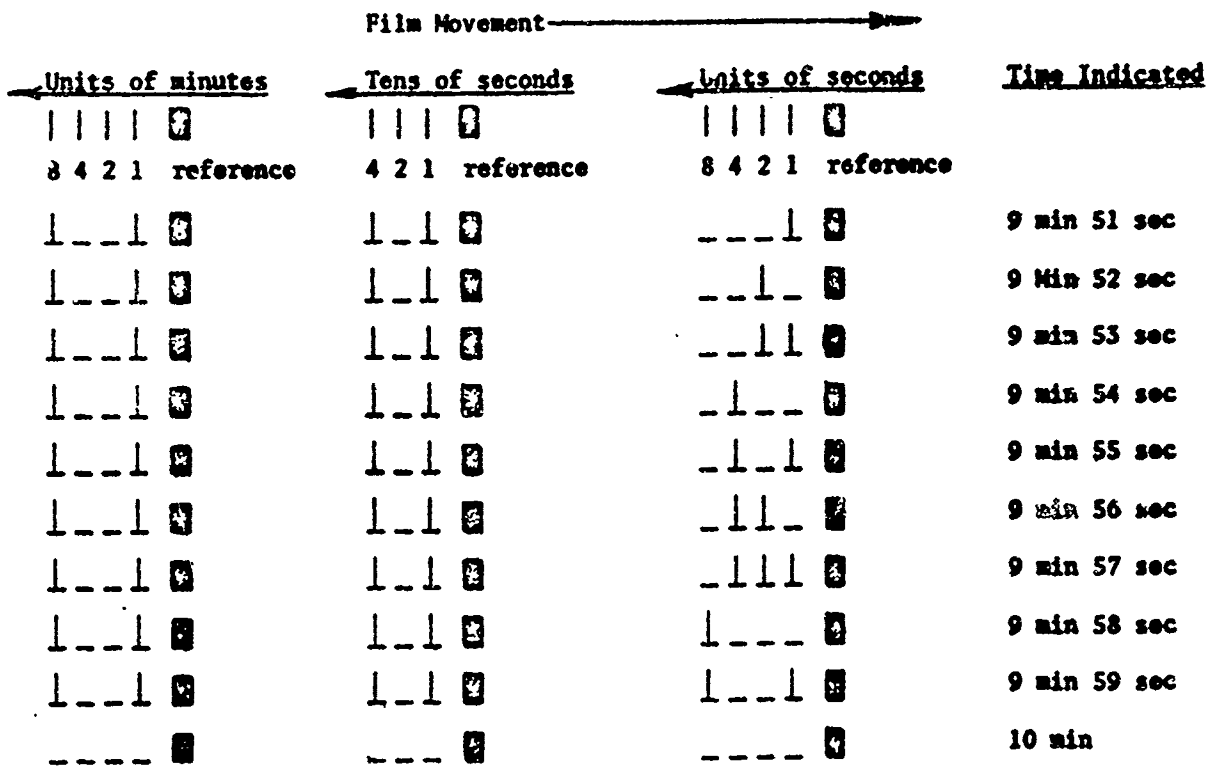


Fig. 18. Timing on Askania Film

Timing on Contraves Film

IRIG Standard Timing (Format B) is used on the Contraves film. Format B is basically a pulse-width code in which the time is expressed twice in each time frame. (Figure 19a)

The first code word is time-of-year expressed as binary coded decimal. The second word is time-of-day expressed in straight binary notation, as seconds only.

Outline of Format B

1. Time: Universal Time (UT-2)

2. Time Frame: 1.0 seconds

3. Code Digit Weighting options: BCD, SB or both

   a. Binary Coded Decimal time-of-year code word

      (1) Seconds, minutes, hours and days

      (2) Recycles yearly

   b. Straight binary time of day code word

      (1) Seconds only

      (2) Recycles each 24 hours

A timing index is used on the Contraves film. It is found at the beginning of the film. It is measured as follows:

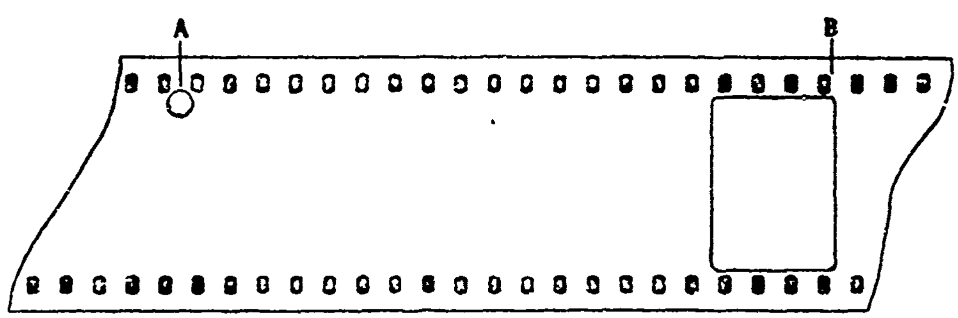
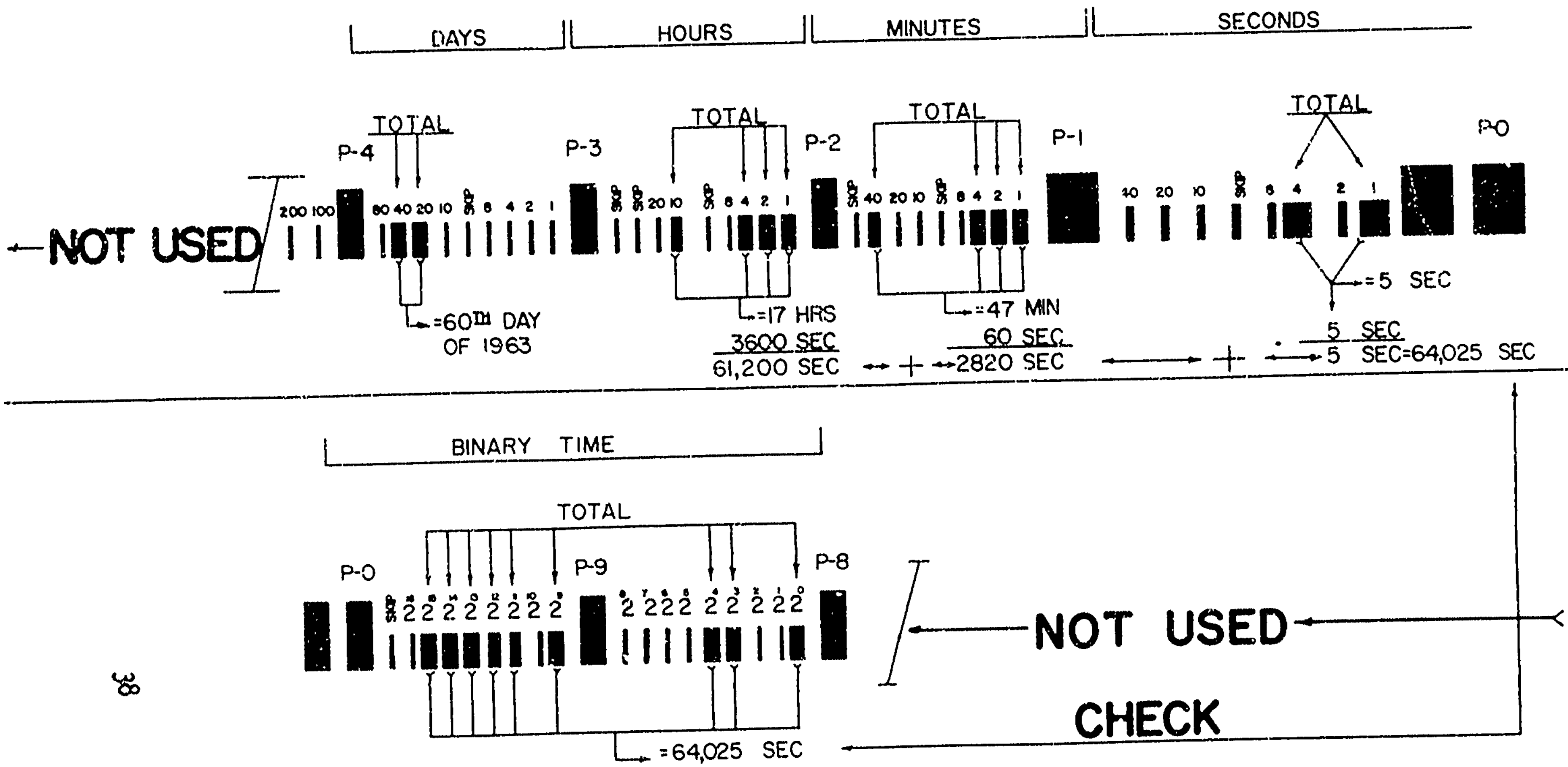


Fig. 19. Timing Index on Contraves Film

The distance AB is measured as shown in Figure 19. When reading the time of the first frame the true time of that frame is located a distance AB from the time actually shown for the frame. The difference between these two times is the correction to be applied to all succeeding frames.

FORMAT-B


DAYS
HOURS
MINUTES
SECONDS
TOTAL
P-4
P-3
P-2
P-1
P-0
NOT USED
200 100
80 40 20 10
8 4 2 1
20 10
40 20 10
=60TH DAY OF 1963
=17 HRS
3600 SEC
61,200 SEC
=47 MIN
60 SEC
2820 SEC
=5 SEC
5 SEC
5 SEC=64,025 SEC
BINARY TIME
P-9
P-8
=64,025 SEC
NOT USED
CHECK

THE CINETHEODOLITE REDUCTION

The data obtained from each cinetheodolite consists of an azimuth and an elevation angle to the missile from the cinetheodolite. The azimuth angle (α) is defined as the angle (measured clockwise from north) of the projection of the missile on the horizontal plane. The elevation angle (ε) is defined as the angle between the line of sight to the missile and the horizontal plane (See Fig. 20).

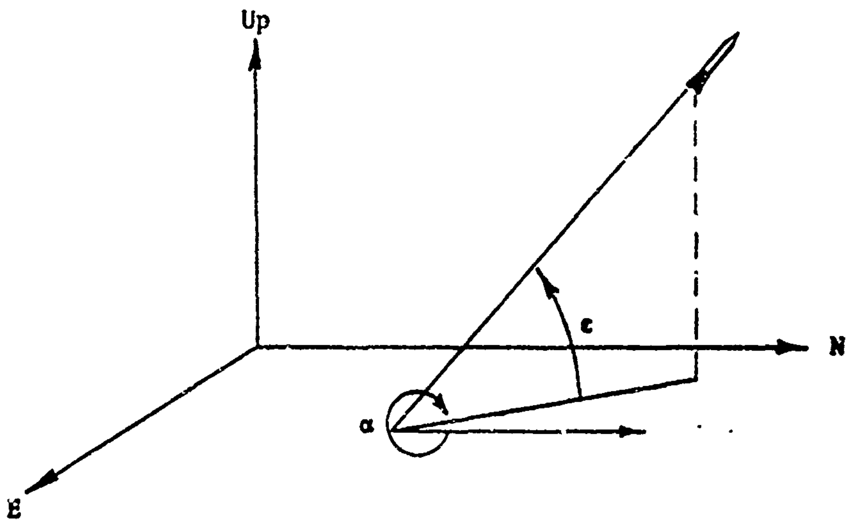


Fig. 20. Cinetheodolite Angles

These measured angles contain errors, some of which are known and some of which are indeterminate. The known errors may be corrected for; the indeterminate errors are minimized by the least squares position solution.

Corrections for known errors in the measured angles or dial readings may be considered as belonging to two groups: Corrections for errors inherent in the camera due to mechanical limitations; and corrections to bring the internal system of the camera into alignment with the external system of the range.

The first group will include eccentricity and lens sag; the second group will include reference zero, collimation, and mislevel (or tilt) and corrections which must be applied to compensate for tracking errors. A further correction must be applied to the position data to compensate for errors caused by refraction.

The following method is used at White Sands Missile Range to determine these corrections. Positioned about each theodolite station are eight equally spaced optical targets. The exact location of these targets has been surveyed. Immediately before and after each missile firing, orientation shots of the targets are taken, first with the camera in a normal position, and then with the camera in a dumped position (elevated through 180 degrees). It is possible to compute the values of corrections for that particular missile firing with data obtained from these orientation shots. These corrections are then applied to the measured angles to obtain the corrected angles for the final cinetheodolite reduction.

Because of the presence of unknown sources of error in the instrument and errors introduced in reading the film, the corrected rays from all the theodolites will be concurrent only with zero probability. Therefore, the problem arises of estimating the position of a missile at a given time from observations yielding a set of non-concurrent lines in space. This is most frequently solved by assuming that the position of the missile is at that point in space which minimizes the sum of the squares of certain residuals (sometimes distances and sometimes angles). The method developed here (the Davis Solution)* is based on the theory of least-squares, minimizing the sum of the squares of the angular residuals. It will be noted that this solution is identical with the maximum likelihood estimates of missile position in the particular case in which the azimuth and elevation angles are normally distributed.

ORIENTATION CALCULATIONS

Scale Factor of Film Reader (Cxy)

$$Cxy = \frac{(180^\circ)(\text{Diagonal of film in inches})}{(\pi)(\text{Focal length in inches})(\text{Diagonal in machine counts})}$$

Theodolite Dial Eccentricity ($\Delta\alpha_E$, $\Delta\varepsilon_E$)

C_g represents the geometric center of the circle. If the dial rotates around another point C_r, the arcs measured on the edge of the circle do not correspond to the turned angles of the dial plate. Eccentricity in the dials may be expressed in terms of:

E = The displacement of the geometric center of the dial (C_g) from the center of rotation of the dial (C_r).

*R.C. Davis, "Techniques for the Statistical Analysis of Cinetheodolite Data," NAVORD RPT 1299, NOTS 369, Naval Ordnance Test Station, China Lake, California, 22 March 1951.

ϕ = angle with respect to reference at which the eccentricity error is increasing through zero.

Figure 21 is the azimuth dial on a theodolite illustrating the following properties:

α_{dio}, ε_{dio} - actual dial readings of the azimuth and elevation dials

ϕ_{α}, ϕ_{ε} - phase angle for azimuth and elevation dials

E_{α}, E_{ε} - magnitude of displacement of center of rotation from geometric center of azimuth and elevation dials.

$\Delta\alpha_E$, $\Delta\varepsilon_E$ - Eccentricity corrections

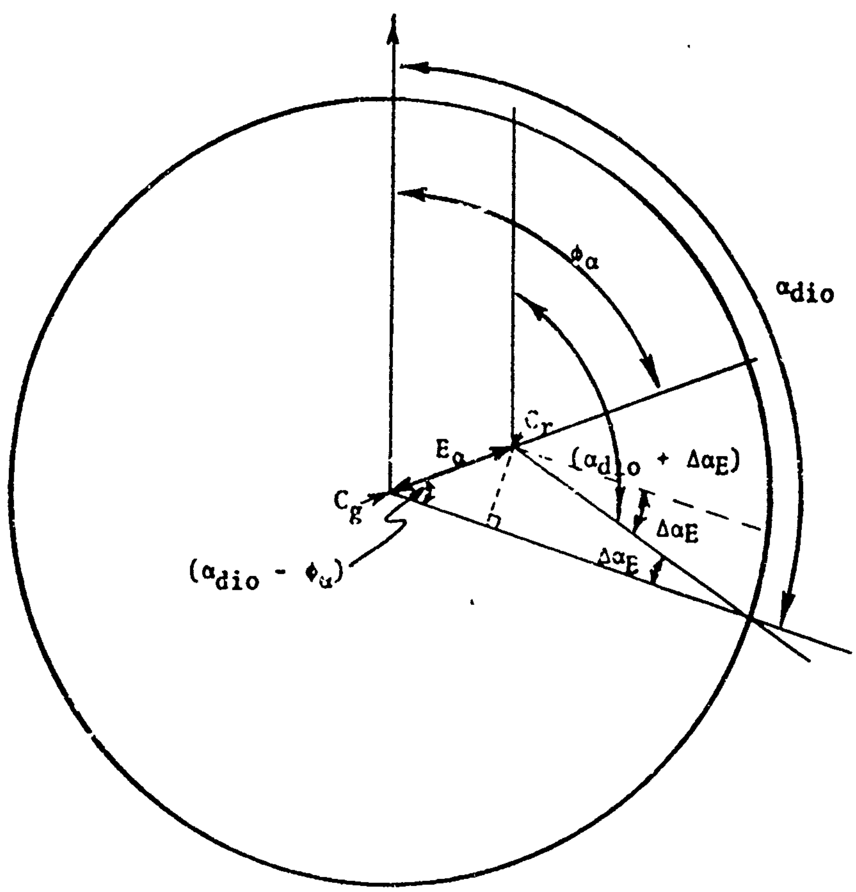


Fig. 21. Dial Eccentricity Correction

$$\tan \Delta\alpha_E = \frac{E_\alpha \sin (\alpha_{dio} - \phi_\alpha)}{1 - E_\alpha \cos (\alpha_{dio} - \phi_\alpha)}$$

$$\tan \Delta\alpha_E \approx E_\alpha \sin (\alpha_{dio} - \phi_\alpha)$$

Since $\Delta\alpha_E$ is a very small angle,

$$\Delta\alpha_E \approx \tan \Delta\alpha_E = E_\alpha \sin (\alpha_{dio} - \phi_\alpha)$$

The same procedure is used to measure the eccentricity in the elevation dial.

Azimuth Zero Correction ($\Delta\alpha_O$) Orientation

Orientation error in azimuth ($\Delta\alpha_O$) is defined as the constant difference between the surveyed and the measured location of the target (within the desired angular reference system).

α_{dio} = dial readings of the frame which shows the i^{th} target board with the camera in a normal position.

X_{di} = boresight measurement corresponding to α_{dio}, in reader counts.

α_{rio} = dial readings of the frame which shows the i^{th} target board with the camera in reversed position.

X_{ri} = boresight measurements corresponding to the α_{rio}, in reader counts.

C_{xy} = degrees count for the reader used to measure X_{di}, X_{ri}.

$\Delta\alpha_E$ = azimuth dial eccentricity correction

α_{si} = surveyed azimuth angle from nodal point of camera to i^{th} target board.

$\alpha_{di} = \alpha_{dio} + (X_{di})\, C_{xy} + \Delta\alpha_E$

$\alpha_{ri} = \alpha_{rio} - (X_{ri})\, C_{xy} - \Delta\alpha_E$

$$\Delta\alpha_i = \alpha_{si} - \frac{1}{2}(\alpha_{di} + \alpha_{ri}) + 90^\circ$$

$$\Delta\alpha_o = \frac{\sum_{i=1}^{N} \Delta\alpha_i}{N}$$

An elevation zero correction, $\Delta\varepsilon_o$, is included in the level correction computation.

<u>Collimation (C_o)</u>

Collimation is defined as the misalignment of the optical and mechanical axes of the camera when the camera is elevated 180 degrees or dumped. The error caused by this misalignment is shown below.

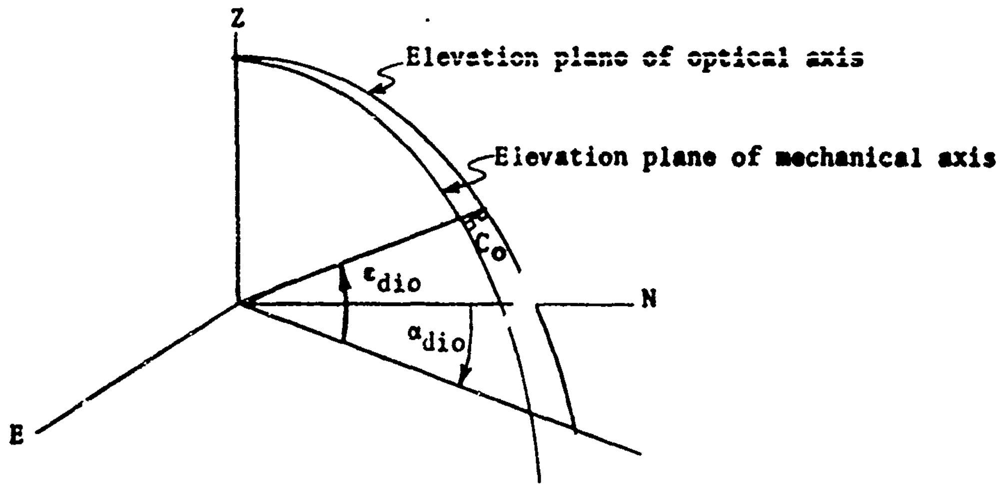


Fig. 22. Collimation Error Correction

α_{di} = azimuth dial reading of the frame which shows the i^{th} target board with camera in normal position.

α_{ri} = azimuth dial reading of the frame which shows the i^{th} target board with camera in reversed position.

$$C_i = \frac{1}{2}(\alpha_{ri} - \alpha_{di}) \pm 90^\circ$$

$$C_o = \frac{\sum_{i=1}^{N} C_i}{N}$$

Lens Sag (D)

Lens sag is caused by non-rigidity of the camera lens barrel. This effect is shown by an exaggerated illustration. Error caused by lens sag is maximum at 0° and is zero at 90°.

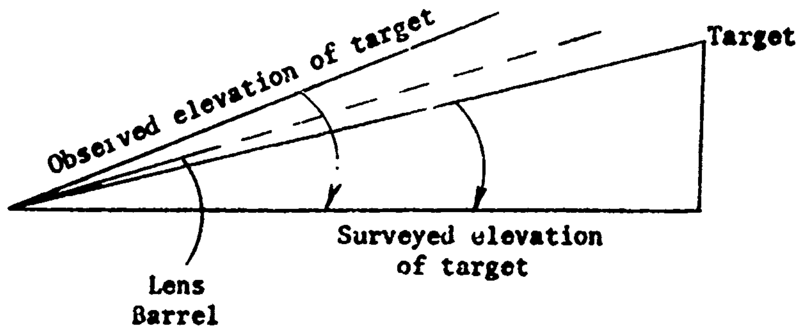


Fig. 23. Lens Sag Error

Then if:

ε_d, ε_r = corrected elevation dial reading of the frame which shows the i^{th} target with camera in normal and reversed position respectively,

$$D = \frac{\varepsilon_d - (\varepsilon_r - 180°)}{2}$$

Level Corrections (L, ϕ_L)

When the camera is not levelled exactly prior to operations, error is introduced as shown in Figure 24.

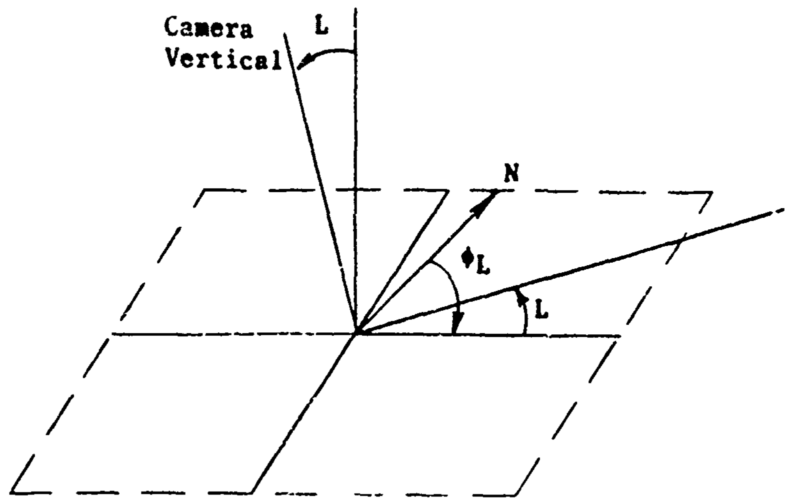


Fig. 24. Level Corrections

This error may be expressed in terms of L, the angle of maximum tilt, and ϕ_L, the phase angle in the azimuth plane at maximum tilt.

Let:

α_{si}, ϵ_{si} = surveyed azimuth and elevation angles from the nodal point of the camera to the ith target.

ϵ_{di} = elevation dial reading of the frame which shows the ith target in normal position.

L = magnitude of level correction

ϕ_L = phase angle

Assume that the error equation is of the form:

$$\Delta\epsilon_i = \Delta\epsilon_o + L \cos (\alpha_{si} - \phi_L)$$

$$= \Delta\epsilon_o + L \sin \alpha_{si} \sin \phi_L + L \cos \alpha_{si} \cos \phi_L$$

where

$$\Delta\epsilon_i = \epsilon_{si} - \epsilon_{di}$$

Using:

$$N = L \cos \phi_L$$

$$M = L \sin \phi_L$$

$$\Delta\varepsilon_i = \Delta\varepsilon_o + M \sin \alpha_{s_i} + N \cos \alpha_{s_i}$$

Using a least squares solution, the following matrices are formed:

$$\begin{pmatrix} N & \Sigma \sin \alpha_{s_i} & \Sigma \cos \alpha_{s_i} \\ \Sigma \sin \alpha_{s_i} & \Sigma \sin^2 \alpha_{s_i} & \Sigma \sin \alpha_{s_i} \cos \alpha_{s_i} \\ \Sigma \cos \alpha_{s_i} & \Sigma \sin \alpha_{s_i} \cos \alpha_{s_i} & \Sigma \cos^2 \alpha_{s_i} \end{pmatrix} \begin{pmatrix} \Delta\varepsilon_o \\ M \\ N \end{pmatrix} = \begin{pmatrix} \Sigma\Delta\varepsilon_i \\ \Sigma\Delta\varepsilon_i \sin \alpha_{s_i} \\ \Sigma\Delta\varepsilon_i \cos \alpha_{s_i} \end{pmatrix}$$

and solved for $\Delta\varepsilon_o$, M, and N.

ϕ_L and L are obtained from:

$$\phi_L = \tan^{-1} \left(\frac{M}{N}\right)$$

$$L = \frac{N}{\cos \phi_L}$$

$\Delta\varepsilon_o$ is the elevation zeroing correction.

<u>Summary of Data Computed for Orientation Calculations</u>

Cxy = Scale Factor

$\Delta\alpha_E$, $\Delta\varepsilon_E$ = Eccentricity

$\Delta\alpha_o$, $\Delta\varepsilon_o$ = Azimuth and elevation zero correction (reference)

C_o = collimation

D = lens sag

L, ϕ_L = magnitude and phase angle of tilt

Corrected Angle Calculations

Given dial reading (α_{dial}, ε_{dial}) and correction from above orientation calculations, find correct α_i and ε_i to be used in the reduction for space positions.

Measured azimuth, elevation:

$$\alpha_{meas} = \alpha_{dial} + \frac{a}{2b} \text{ (See Fig. 8)}$$

$$\varepsilon_{meas} = \varepsilon_{dial} + \frac{a}{2b} \text{ (See Fig 8)}$$

Eccentricity correction

$$\alpha_E = \alpha_{meas} + \Delta\alpha_E$$

$$\varepsilon_E = \varepsilon_{meas} + \Delta\varepsilon_E$$

Reference azimuth, elevation

$$\alpha_o = \alpha_E + \Delta\alpha_o$$

$$\varepsilon_o = \varepsilon_E + \Delta\varepsilon_o$$

Lens sag

$$\varepsilon_D = \varepsilon_o + D \cos \varepsilon_o$$

X and Y reader counts converted to collimation for Askania are

$$\bar{X} + XC_{xy} + C_o$$

$$\bar{Y} = YC_{xy}$$

and Contraves

$$\bar{X} = (-X \cos \varepsilon_D + Y \sin \varepsilon_D)\, C_{xy} + C_o$$

$$\bar{Y} = (X \sin \varepsilon_D + Y \cos \varepsilon_D)\, C_{xy}$$

Then correcting the angles for collimation,

$$\varepsilon_C = \sin^{-1}\left[\frac{\sin \varepsilon_D + \bar{Y} \cos \varepsilon_D}{(1 + \bar{X}^2 + \bar{Y}^2)^{1/2}}\right]$$

$$\alpha_C = \alpha_0 + \sin^{-1}\left[\frac{\bar{X}}{\cos \varepsilon_C \, (1 + \bar{X}^2 + \bar{Y}^2)^{1/2}}\right]$$

<u>Level Correction</u> (L, ϕ_L)

$$(\alpha_i - \phi_L) = \sin^{-1}\left[\frac{\cos \varepsilon_C \sin (\alpha_C - \phi_L)}{\cos \varepsilon_L}\right]$$

$$\alpha_i = \phi_L \cos^{-1}\left(\frac{\sin \varepsilon_L \cos L - \sin \varepsilon_C}{\cos \varepsilon_L \sin L}\right)$$

$$\varepsilon_L = \sin^{-1}\left[\sin \varepsilon_C \cos L + \cos \varepsilon_C \sin L \cos (\alpha_C - \phi_L)\right]$$

$$= \cos^{-1}\left[(1 - \sin^2 \varepsilon_L)^{1/2}\right]$$

<u>Refraction Correction</u>

The exact formulas for the refraction effect are extremely complicated. Of the various approximate formulas which have been proposed, the following has been found to fit the rigorous curve within adequate accuracy over the limits in which it is used in cinetheodolite reductions.

$$\Delta\varepsilon_R = \frac{-0.023157\,(X^2 + Y^2)^{1/2}}{Z + 100{,}000}$$

Where $(X^2 + Y^2)^{1/2}$ = distance in feet to the projection of the missile in the horizontal plane, and Z = distance in feet to the missile above the horizontal plane, $\Delta\varepsilon_R$ is the correction in degrees which must be applied to the measured elevation. An approximate position of the missile is required to make this correction.

COMPUTATIONAL PROCEDURE

The coordinate system used is a cartesian system (X, Y, Z) where X is positive to the North, Y is positive to the East and Z is positive upwards from the origin.

Using two stations an approximate position (X_o, Y_o, Z_o) is computed by the Bodwell* method. If any one coordinate disagreement is greater than 300 feet another position is computed using one of the first two stations and a third station. If there is still a disagreement of over 300 feet then the second and third stations are used to compute the approximate position.

Assuming that X_o, Y_o, Z_o is a close approximation, the elevation corrected for refraction may be computed by

$$\varepsilon_i = \varepsilon_L + \Delta\varepsilon_R$$

A Bodwell point is used as the initial point for each position throughout the missile run. The Davis least square solution is then used to compute ΔX, ΔY, and ΔZ as follows:

A first approximation to the azimuth and elevation angles from the i^{th} theodolite is computed.

$$\alpha_i^* = \tan^{-1}\left(\frac{Y_o - Y_i}{X_o - X_i}\right)$$

$$\varepsilon_i^* = \tan^{-1} \frac{Z_o - Z_i}{\sqrt{(X_o - X_i)^2 + (Y_o \quad Y_i)^2}}$$

where X_i, Y_i, Z_i are the coordinates of the i^{th} theodolite.

*See A Least Squares Solution of the Cinetheodolite Problem, C.A. Bodwell, HADC Report No. MTHT-138, 12 Dec. 1951

The true azimuth and elevation from the station are expressed in a Taylor's series expansion in which the higher order terms are discarded as negligible. These series are:

$$\alpha_i = \alpha^{*}_i + \frac{\partial \alpha^{*}_i}{\partial x} \Delta x_i + \frac{\partial \alpha^{*}_i}{\partial y} \Delta y_i + \frac{\partial \alpha^{*}_i}{\partial z} \Delta z_i + \cdots \tag{1}$$

$$\varepsilon_i = \varepsilon^{*}_i + \frac{\partial \varepsilon^{*}_i}{\partial x} \Delta x_i + \frac{\partial \varepsilon^{*}_i}{\partial y} \Delta y_i + \frac{\partial \varepsilon^{*}_i}{\partial z} \Delta z_i + \cdots \tag{2}$$

The function to be minimized is the sum of the squares of the residuals. The residuals are the changes in the angles. This sum becomes:

$$s = \sum_{i=1}^{N} \left[\left(\alpha_i - \alpha^{*}_i - \frac{\partial \alpha^{*}_i}{\partial x} \Delta x_i - \frac{\partial \alpha^{*}_i}{\partial y} \Delta y_i - \frac{\partial \alpha^{*}_i}{\partial z} \Delta z_i \right)^2 P_i + \left(\varepsilon_i - \varepsilon^{*}_i \right.\right.$$

$$\left.\left. - \frac{\partial \varepsilon^{*}_i}{\partial x} \Delta x_i - \frac{\partial \varepsilon^{*}_i}{\partial y} \Delta y_i - \frac{\partial \varepsilon^{*}_i}{\partial z} \Delta z_i \right)^2 Q_i \right] \tag{3}$$

where P_i and Q_i are the angular weighting functions defined by:

$$P_i = \cos^2 \varepsilon_i = \frac{(X_o - X_i)^2 + (Y_o - Y_i)^2}{(X_o - X_i)^2 + (Y_o - Y_i)^2 + (Z_o - Z_i)^2}$$

$$Q_i = 1$$

Substituting: $P_i a_{i1} = \dfrac{\partial \alpha_i^0}{\partial x} P_i$

$$P_i a_{i2} = \frac{\partial \alpha_i^0}{\partial y} P_i$$

$$P_i a_{i3} = \frac{\partial \alpha_i^0}{\partial z} P_i$$

$$Q_i e_{i1} = \frac{\partial \varepsilon_i^0}{\partial x} Q_i$$

$$Q_i e_{i2} = \frac{\partial \varepsilon_i^0}{\partial y} Q_i$$

$$Q_i e_{i3} = \frac{\partial \varepsilon_i^0}{\partial z} Q_i \qquad (4)$$

and the approximate positions with respect to the i^{th} theodolite,

$$X_o - X_i = x_i$$

$$Y_o - Y_i = y_i$$

$$Z_o - Z_i = z_i$$

and letting

$$g_{kj} = g_{jk} = \sum_{i=1}^{N} a_{ij}\, a_{ik}\, P_i$$

$$h_{kj} = h_{jk} = \sum_{i=1}^{N} e_{ij}\, e_{ik}\, Q_i$$

Express the simultaneous equations represented by

$$\frac{\partial s}{\partial x} = 0$$

$$\frac{\partial s}{\partial y} = 0 \qquad (5)$$

$$\frac{\partial s}{\partial z} = 0$$

in the following form:

$$(g_{11} + h_{11})\,\Delta x_i + (g_{12} + h_{12})\,\Delta y_i + (g_{13} + h_{13})\,\Delta z_i =$$

$$\sum \left[a_{i1}\,(\alpha_i - \alpha^*_i)\,P_i + e_{i1}\,(\varepsilon_i - \varepsilon^*_i)\,Q_i\right] \qquad (6)$$

$$(g_{12} + h_{12})\,\Delta x_i + (g_{22} + h_{22})\,\Delta y_i + (g_{23} + h_{23})\,\Delta z_i =$$

$$\sum \left[a_{i2}\,(\alpha_i - \alpha^*_i)\,P_i + e_{i2}\,(\varepsilon_i - \varepsilon^*_i)\,Q_i\right] \qquad (7)$$

$$(g_{13} + h_{13})\,\Delta x_i + (g_{23} + h_{23})\,\Delta y_i + (g_{33} + h_{33})\,\Delta z_i =$$

$$\sum \left[a_{i3}\,(\alpha_i - \alpha^*_i)\,P_i + e_{i3}\,(\varepsilon_i - \varepsilon^*_i)\,Q_i\right] \qquad (8)$$

The values of the coefficients of the variables Δx_i, Δy_i, and Δz_i in Equations 6-8, evaluated at the first approximation, using Equation 4 where

$$R_i^2 = x_i^2 + y_i^2 + z_i^2$$

become:

$$g_{11} + h_{11} = \sum_{i=1}^{N} \frac{y_i^2 + z_i^2}{R_i^4}$$

$$g_{12} + h_{12} = \sum_{i=1}^{N} - \left(\frac{x_i\, y_i}{R_i^4}\right)$$

$$g_{13} + h_{13} = \sum_{i=1}^{N} - \left(\frac{x_i \, z_i}{R_i^{\,4}}\right)$$

$$g_{23} + h_{23} = \sum_{i=1}^{N} - \left(\frac{y_i \, z_i}{R_i^{\,4}}\right)$$

$$g_{22} + h_{22} = \sum_{i=1}^{N} \frac{x_i^{\,2} + z_i^{\,2}}{R_i^{\,4}}$$

$$g_{33} + h_{33} = \sum_{i=1}^{N} \frac{x_i^{\,2} + y_i^{\,2}}{R_i^{\,4}}$$

by substituting

$$A_i = \frac{x_i}{R_i^{\,2}} \tag{9}$$

$$B_i = \frac{y_i}{R_i^{\,2}} \tag{10}$$

$$C_i = \frac{z_i}{R_i^{\,2}} \tag{11}$$

express the coefficients of the variables in Equations 6-8 as:

$$\Delta = \begin{vmatrix} \sum\limits_{i=1}^{N} (B_i^2 + C_i^2) & -\sum\limits_{i=1}^{N} A_i B_i & -\sum\limits_{i=1}^{N} A_i C_i \\ -\sum\limits_{i=1}^{N} A_i B_i & \sum\limits_{i=1}^{N} (A_i^2 + C_i^2) & -\sum\limits_{i=1}^{N} B_i C_i \\ -\sum\limits_{i=1}^{N} A_i C_i & -\sum\limits_{i=1}^{N} B_i C_i & \sum\limits_{i=1}^{N} (A_i^2 + B_i^2) \end{vmatrix} \tag{12}$$

and the cofactors Δ_{11}, Δ_{12}, Δ_{13}, Δ_{22}, Δ_{23}, Δ_{33}, $(\Delta_{jk} = \Delta_{kj})$ of the determinant Δ may be readily obtained. The space positions may be computed from the solution of the simultaneous Equations 6-8, yielding:

$$\Delta x_i = \sum_{i=1}^{N} \left[a_{i1} (\alpha_i - \alpha^\circ_i) P_i + e_{i1} (\varepsilon_i - \varepsilon^\circ_i) Q_i \right] \frac{\Delta_{11}}{\Delta}$$

$$+ \sum_{i=1}^{N} \left[a_{i2} (\alpha_i - \alpha^\circ_i) P_i + e_{i2} (\varepsilon_i - \varepsilon^\circ_i) Q_i \right] \frac{\Delta_{12}}{\Delta} \qquad (13)$$

$$+ \sum_{i=1}^{N} \left[a_{i3} (\alpha_i - \alpha^\circ_i) P_i + e_{i3} (\varepsilon_i - \varepsilon^\circ_i) Q_i \right] \frac{\Delta_{13}}{\Delta}$$

with similar expressions for Δy_i and Δz_i.

The new approximate position then becomes

$$X_{Ai} = X_o + \Delta x_i$$

The new estimated point is then used to find a new Δx, Δy, Δz. This procedure (iteration) is continued either until Δx, Δy and Δz are less than 0.1 units or through six iterations.

The azimuth and elevation residuals for each station are computed (in radians). The azimuth residual is multiplied by the cosine of the elevation. After the last iteration, if the absolute value of any one azimuth or elevation residual exceeds 2.25 minutes, the azimuth and elevation angles for that station are removed and the point is recomputed.

VARIANCE AND CO-VARIANCE OF THE COORDINATES

The angular standard deviations for each point may be found from the residual angles ($\delta\alpha_i \cos \varepsilon_i$ and $\delta\varepsilon_i$) using

$$\sigma A = \sqrt{\frac{\Sigma[(\delta\alpha_i) \cos \varepsilon_i]^2 + \Sigma\delta\varepsilon^2{}_i}{2n-3}}$$

The coordinate σ's are then computed using the σA above and the cofactors of the elements of the principal diagonal of the least squares determinant.

$$\sigma_x = \sqrt{\frac{\Delta_{11}\ \sigma_A{}^2}{\Delta}}$$

$$\sigma_y = \sqrt{\frac{\Delta_{22}\ \sigma_A{}^2}{\Delta}}$$

$$\sigma_z = \sqrt{\frac{\Delta_{33}\ \sigma_A{}^2}{\Delta}}$$

Rotational Computations

The position data, having been computed in the WSCS, are translated to the desired origin and rotated into the desired tangent plane.

Hand Computing Space Position

A simple two-station reduction (Bodwell Solution) may be used to check the data which is received from the high speed computer, IBM 7094, starting with the corrected angles, α_1, ε_1, α_2, ε_2, as follows:

1. $\cos \alpha_1$
2. $\cos \varepsilon_1$
3. $\sin \alpha_1$
4. $\cos \alpha_2$
5. $\cos \varepsilon_2$
6. $\sin \alpha_2$
7. $a_1 = \cos \alpha_1 \cos \varepsilon_1$
8. $a_2 = \cos \alpha_2 \cos \varepsilon_2$
9. $c_1 = \sin \varepsilon_1$
10. $c_2 = \sin \varepsilon_2$
11. $b_1 = \cos \varepsilon_1 \sin \alpha_1$
12. $b_2 = \cos \varepsilon_2 \sin \alpha_2$
13. $A = a_1 a_2 + b_1 b_2 + c_1 c_2$
14. $B_1 = (x_2 - x_1)\, a_1 + (y_2 - y_1)\, b_1 + (z_2 - z_1)\, c_1$
15. $B_2 = (x_2 - x_1)\, a_2 + (y_2 - y_1)\, b_2 + (z_2 - z_1)\, c_2$
16. $A^2 - 1$
17. $r_1 = \dfrac{B_2 A - B_1}{A^2 - 1}$
18. $r_2 = \dfrac{-B_1 A + B_2}{A^2 - 1}$
19. $x_{01} = r_1 a_1 + x_1$
20. $y_{01} = r_1 b_1 + y_1$
21. $z_{01} = r_1 c_1 + z_1$
22. $x_{02} = r_2 a_2 + x_2$

23. $y_{o2} = r_2 b_2 + y_2$

24. $z_{o2} = r_2 c_2 + z_2$

25. $$x_o = \frac{x_{o1} + x_{o2}}{2}$$

26. $$y_o = \frac{y_{o1} + y_{o2}}{2}$$

27. $$z_o = \frac{z_{o1} + z_{o2}}{2}$$

(where x_1, y_1, z_1, x_2, y_2, z_2 are the surveyed coordinates of the camera stations, translated and rotated.)

CINETHEODOLITE REFERENCES

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EOTS EQUIPMENT, Contraves AG, Zurich, Switzerland

IRIG STANDARD TIME FORMATS, Document Nr. 104-60, IRIG, White Sands Missile Range, New Mexico, 1960

B. POSITION DATA

II Single Station AN/FPS-16 Radar, including Derivation of Dew Point Temperature

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WSMR SINGLE STATION AN/FPS-16 RADAR REDUCTION

Table of Contents

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INTRODUCTION

The primary electronic tracking instrumentation system at WSMR is the AN/FPS-16 Radar system. The system consists of nine radars, located throughout the range: three at "C" station, two each at King-I and Stallion site, and one each at Tula Range and Phillips Hill.

Although often referred to as a "system", it is actually simply a group of individual instruments, since each radar is capable of independently determining the position of the object being tracked. The radar reduction discussed in this section is a single-station solution for determining the trajectory of a missile (or other target) in cartesian coordinates using the range, azimuth, and elevation data as measured by one radar station. An N-station solution which determines a single trajectory from the weighted averages of multiple-radar measurements will be discussed in a later section.

Basically a radar operates by transmitting a high-energy radio frequency signal. A portion of this signal is intercepted by a reflecting object (target) and reradiated in all directions. That energy which is reradiated back to the radar is detected by the radar receiver. The distance to the target is determined from the time elapsed between the transmission of the signal and the detection of the echo. The direction or angular position of the target is indicated by the pointing angles of the antenna. (Normally a radar uses the same antenna for both transmitting and receiving.)

The most common form of rf signal transmitted is a train of narrow rectangular pulses, modulating a sine wave carrier. Since electromagnetic energy travels at the speed of light, the range to the target can be found from $R = \frac{c\,\Delta t}{2}$ where c is the velocity of light, and Δt is the time required for the wave to travel out and back.

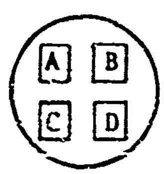
Once the transmitted pulse is emitted, a sufficient length of time must be permitted to elapse in order that any return echo may be detected before the next pulse is transmitted. (Otherwise the Δt would be measured with respect to the wrong transmitted pulse and erroneous range data produced.) The measurement of range becomes simply a problem of time measurement. The pulse repetition frequency (prf), i.e., the rate at which pulses are transmitted, determines the maximum range from which echoes can be returned without ambiguity:

$$R_{(\text{max. unambiguous})} = \frac{c}{2(\text{PRF})}$$

The radar transmits a narrow beam of radiation, using a highly directional antenna. Consequently echoes are received only from targets lying in the direction in which the beam is pointing. If the antenna, and therefore the radar beam, is swept or scanned around the horizon, the strongest echo will be received when the beam is pointed directly at the target, weaker echoes when the beam is pointed a little to one side or another of it, and no echo when it is pointing in other directions. By measuring the antenna position (in azimuth and elevation angles) at the time the strongest echo is received, the position of the target can be determined.

A tracking radar is designed so that, once it has located the target which is to be tracked, it will "lock on" and automatically continue to point its antenna in the proper direction to follow the trajectory of the target. The FPS-16's are equipped with a closed loop servo control system to perform this function.

The FPS-16 is a monopulse tracking radar, having four feed horns located at the focal point of its parabolic antenna. If the target is centered directly in the beam path, equal amounts of returned radiation will be received by each of the four horns. If, however, the target is off-center, by comparisons of the signal strength received at each of the horns, error signals can be developed to direct the servo system to automatically correct the antenna pointing direction. The azimuth tracking error signal is proportional to the difference between the sums of each vertical pair of horns. The elevation tracking error signal is proportional to the difference between the sums of the horizontal pairs. (See Fig. 1)



$$\Delta Az. = (A+C) - (B+D)$$

$$\Delta El. = (A+B) - (C+D)$$

FIGURE 1 - Feed Horn Configuration

The sum of the radiation received by all four horns is used to develop the reference signal to which the azimuth and elevation error signals are related. This sum signal is also used to develop the automatic gain control (AGC) signal. The AGC controls the amplifier gain of the radar receiver to insure that the signal level in the receiver is kept within the limits of the receiver's linearity, and to prevent losses of small signals by noise or large signals by receiver saturation. (Another application of this AGC signal will be discussed in connection with the "Radar Cross-Section" data reduction.)

The basic data obtained by a radar consist of range, azimuth, and elevation observations, with timing recorded digitally on magnetic tape in binary code. The FPS-16's at WSMR use standard IRIG timing.

AN/FPS-16 RADAR DATA CORRECTION

A precise calibration technique is necessary to correct the FPS-16 raw data for known errors. This includes orientation calibrations, data shaft eccentricity corrections, mislevel and antenna sag corrections, beacon delay corrections, and refraction corrections.

Orientation Calibrations

Each AN/FPS-16 radar employed at White Sands Missile Range utilizes an orientation system of six orientation targets, boresight signal generator, range corner reflector and two boresight targets. The targets are referenced in elevation and azimuth with respect to a local grid system and positioned 500 yards from the mechanical axis of the radar. The target boards are four feet square and are mounted on sixteen foot poles approximately eight feet above the ground. Each target is quartered and brightly colored with opposite quarters of the same color. One target, the grid target, is positioned in the same horizontal plane of the radar at an elevation equal to zero mils to insure an elevation plunge (dump) angle of exactly 180 degrees (3200 mils).

An optical telescope is used to sight the targets for positioning the radar. The telescope can be moved from its mount. The telescope mount is welded on the antenna support structure of the radar.

The boresight signal generator and the boresight targets are mounted on the boresight tower approximately five hundred yards from the radar.

All calibration data are supplied by the radar division to Data Reduction on a data correction sheet (Fig. 2).

The first step of the calibration procedure is to determine the relationship of the vertical plane through the optical axis of the optical telescope to the vertical plane which is perpendicular to the elevation axis of the radar. This is done in the following manner:

a. Place the telescope in its mount in the reverse position while the antenna is in the normal position.

b. Plunge the antenna in elevation and direct the mount until the vertical cross hair of the telescope coincides exactly with the vertical line on the grid target. (Read azimuth octal at the console.)

c. Return the antenna to its normal position and return the telescope to its normal position. (Note that the azimuth reading has not changed.)

d. Read the deviation of the grid target board vertical cross hair with respect to the telescope vertical cross hair. The sign is positive if the grid target cross hair is to the right of the telescope cross hair, negative if to the left. The deviation (c_1) is recorded on the data correction sheet.

The next step in the procedure is to determine the location of the beam axis with respect to the plane which is perpendicular to the elevation axis of the radar. This is done by:

a. Track the boresight tower in the automatic mode at the receiver frequency and receiver bandwidth that is to be used during the mission, with the mount and optical telescope in the normal position.

b. Observe the optical target on the boresight tower with the optical telescope. Read "a_1", as positive if the vertical line of the target is to the right of the vertical cross hair of the optical telescope, negative if to the left. Read "e_1", as positive if the horizontal line of the target is above the horizontal cross hair of the optical telescope, negative if it is below. The "a_1", and "e_1", are recorded on the data correction sheet.

c. Plunge the antenna in elevation and rotate 3200 mils in azimuth, and track the boresight tower again. Read "a_2", and "e_2" as above and record them on the correction sheet.

d. Observe the six orientation targets in sequence. Direct the mount until the horizontal and vertical cross hairs of the telescope coincide with the cross hairs of the targets. Record on the correction sheet the azimuth and elevation readings converted to mils for each target.

The last step in the calibration procedure is to determine the range calibration.

a. The radar tracks the corner reflector, the signal in the skin gate. The octal range reading is converted to yards and recorded on the data sheet.

Data Shaft Eccentricities

The data shaft eccentricity error is defined as the error introduced in the recorded data by the data shafts' not rotating about their true centers. The eccentricity constants for each radar are supplied to Data Reduction by the Radar section.

Beacon Delay Correction

The time delay between the reception of the radar signal by the beacon transponder in the missile and the transmission of the transponder's own signal is known as the beacon delay.

Prior to a test, the beacon delay is measured and compensated for in the field. The field measurement is compared with the beacon and skin data differences. If the differences are not zero, then a correction is added to the field measurement to produce zero differences.

If circumstances prevent the beacon delay setting, the negative of the beacon delay (Q) must be recorded on the data correction sheet.

Other Error Constants

The error constant (H) is that introduced by the elevation axis of the radar not being perpendicular to the azimuth axis.

The error constant (D) is that introduced in the elevation angle due to the sag caused by the weight of the antenna.

Refraction Corrections

Refraction error or propagation error is that error introduced in the elevation angle and range due to the variations in the velocity of propagation through the atmosphere.

The values recorded on the data correction sheet are:

a. The wet bulb (T_W) and dry bulb (T_O) temperatures which are determined by the use of a psychrometer.

b. The barometric pressure (P_O).

Mislevel Correction

The mislevel error is that error introduced by the azimuth plane of the radar not being coplanar with the surveyed azimuth plane.

Definitions of Symbols Used in the Reduction:

a_1, a_2, e_1, e_2 = Optical to beam axis observations

a, e = Corrections for non-perpendicularity of beam axis to elevation axis

C_O = Correction for non-perpendicularity of beam axis to the elevation axis ((Collimation)

c_1 = deviation of grid target vertical cross hair with respect to telescope vertical cross hair

α_{oi} = Recorded azimuth of the i^{th} target

ε_{oi} = Recorded elevation of the i^{th} target

$e_{\varepsilon 1}$ = Amplitude of elevation 1 speed shaft eccentricity

$e_{\varepsilon 16}$ = Amplitude of elevation 16 speed shaft eccentricity

$\phi_{\varepsilon 1}$ = Phase of elevation 1 speed shaft eccentricity

$\phi_{\epsilon 16}$ = Phase of elevation 16 speed shaft eccentricity

$e_{\alpha 1}$ = Amplitude of azimuth 1 speed shaft eccentricity

$e_{\alpha 16}$ = Amplitude of azimuth 16 speed shaft eccentricity

$\phi_{\alpha 1}$ = Phase of azimuth 1 speed shaft eccentritity

$\phi_{\alpha 16}$ = Phase of azimuth 16 speed shaft eccentricity

H = Non-perpendicularity of azimuth axis of the radar to the elevation axis of the radar

D = Error due to sag caused by the weight of the antenna

α_{si} = Surveyed azimuth angle of the i^{th} target

ϵ_{si} = Surveyed elevation angle of the i^{th} target

L = Amplitude of tilt

ϕ_L = Azimuth angle away from true north, at which the maximum mislevel occurs

$\Delta\epsilon_o$ = Elevation calibration correction (zeroing correction)

$\Delta\alpha_o$ = Azimuth calibration correction (zeroing correction)

n = number of targets

e_R = Amplitude of range 1 speed shaft eccentricity

ϕ_R = Phase of range 1 speed shaft eccentricity

R_o = Observed Range

R_s = Surveyed Range

ΔR_c = Range calibration correction

$\delta\epsilon_i$, $\delta\alpha_i$ = Angular residuals for i^{th} target

K_{1e}, K_{2e}, K_{1R}, K_{2R} = Coefficients of refraction

N = Index of refraction (not to be confused with $N = L \cos \phi_L$)

P_o = Pressure in millibars

T_w = Wet bulb temperature in degrees absolute

T_o = Dry bulb temperature in degrees absolute

ε_{oj}, α_{oj} = Observed angles at the j^{th} time

R_{oj} = Observed range at the j^{th} time

Q = Beacon delay

$$F = \frac{\text{free space velocity of propagation}}{\text{Radar velocity of propagation}} = 1.000000$$

ε_{6j}, α_{6j} = Final corrected angles at j^{th} time in the radar tangent plane

R_{6j} = Final corrected range at j^{th} time

λ_j, μ_j, ν_j = Direction cosines of radar line of sight at j^{th} time

X_j'', Y_j'', Z_j'' = Coordinates at j^{th} time in WSCS plane

X_j', Y_j', Z_j' = Coordinates at j^{th} time with respect to desired origin in WSCS plane

X_j, Y_j, Z_j = Coordinates at j^{th} time with respect to tangent plane of desired origin

Mathematical Procedure

The following is the procedure for the single station reduction of AN/FPS-16 radar data.

Angular Orientation Calculations:

The corrections for the error introduced by the radar beam axis not being perpendicular to the elevation axis are computed as follows:

1. $a = \dfrac{a_1 - a_2}{2}$

2. $e = \dfrac{e_1 - e_2}{2}$

3. $C_o = a - \dfrac{c_1}{2}$

By applying corrections (1) and (2) to the recorded azimuth and elevation angles, all observations become referenced to the beam axis.

$\varepsilon_{1i} = \varepsilon_{oi} - e$

$\alpha_{1i} = \alpha_{oi} - a$

The angles are then corrected for data shaft eccentricity errors by the following equations.

$$\epsilon_{2i} = \epsilon_{1i} + e_{\epsilon 1} \sin (\epsilon_{1i} - \phi_{\epsilon 1}) + e_{\epsilon 16} \sin 16 (\epsilon_{1i} - \phi_{\epsilon 16})$$

$$\alpha_{2i} = \alpha_{1i} + e_{\alpha 1} \sin (\alpha_{1i} - \phi_{\alpha 1}) + e_{\alpha 16} \sin 16 (\alpha_{1i} - \phi_{\alpha 16})$$

The angles corrected for the error constant (H), introduced by the non-perpendicularity of the elevation axis to the azimuth axis, become

$$\epsilon_{3i} = \sin^{-1} (\sin \epsilon_{2i} \cos |H|)$$

$$\alpha_{3i} = \alpha_{2i} + \sin^{-1} \left(\sin H \left|\frac{\sin \epsilon_{2i}}{\cos \epsilon_{3i}}\right|\right)$$

Then, correcting for the error constant (D), due to the sag caused by the weight of the antenna, the elevation becomes

$$\epsilon_{4i} = \epsilon_{3i} - D \cos \epsilon_{3i}$$

The azimuth angle is corrected for collimation (C_0), the error due to the non-perpendicularity of the beam axis to the elevation axis, by the following equation:

$$\alpha_{4i} = \alpha_{3i} + C_0 \sec \epsilon_{3i}$$

Using the surveyed angles of the targets and the corrected angles compute

$$\Delta\alpha_i = \alpha_{si} - \alpha_{4i}$$

$$\Delta\epsilon_i = \epsilon_{si} - \epsilon_{4i}$$

The correction for the non-level error, which is due to the azimuth plane of the radar not being coplanar with designated azimuth plane, is computed using the surveyed azimuth data of the targets and the corrected elevation angles of the targets.

Assuming that the error equation is of the form:

$$\epsilon_{si} - \epsilon_{4i} = \Delta\epsilon_0 + L \cos (\alpha_{si} - \phi_L)$$

$$= \Delta\epsilon_0 + L \cos \alpha_{si} \cos \phi_L + L \sin \alpha_{si} \sin \phi_L \qquad (1)$$

let $M = L \sin \phi_L$

$N = L \cos \phi_L$

Since $\Delta\epsilon_i = \epsilon_{si} - \epsilon_{4i}$, equation (1) becomes:

$$\Delta\epsilon_i = \Delta\epsilon_o + M \sin \alpha_{si} + N \cos \alpha_{si}$$

Using a least squares method, the sum to be minimized is:

$$S = \sum_{i=1}^{n} (\Delta\epsilon_o + M \sin \alpha_{si} + N \cos \alpha_{si} - \Delta\epsilon_i)^2$$

and

$$\frac{\partial S}{\partial \Delta\epsilon_o} = 2\left[n\,\Delta\epsilon_o + M \sum_{i=1}^{n} \sin \alpha_{si} + N \sum_{i=1}^{n} \cos \alpha_{si} - \sum_{i=1}^{n} \Delta\epsilon_i\right]$$

$$\frac{\partial S}{\partial M} = 2\left[\Delta\epsilon_o \sum_{i=1}^{n} \sin \alpha_{si} + M \sum_{i=1}^{n} \sin^2 \alpha_{si} + N \sum_{i=1}^{n} \sin \alpha_{si} \cos \alpha_{si} - \sum_{i=1}^{n} \Delta\epsilon_i \sin \alpha_{si}\right]$$

$$\frac{\partial S}{\partial N} = 2\left[\Delta\epsilon_o \sum_{i=1}^{n} \cos \alpha_{si} + M \sum_{i=1}^{n} \sin \alpha_{si} \cos \alpha_{si} + N \sum_{i=1}^{n} \cos^2 \alpha_{si} - \sum_{i=1}^{n} \Delta\epsilon_i \cos \alpha_{si}\right]$$

The simultaneous equations are represented by

$$n\,\Delta\varepsilon_o + N\sum_{i=1}^{n}\cos\alpha_{si} + M\sum_{i=1}^{n}\sin\alpha_{si} = \sum_{i=1}^{n}\Delta\varepsilon_i$$

$$\Delta\varepsilon_o\sum_{i=1}^{n}\sin\alpha_{si} + N\sum_{i=1}^{n}\sin\alpha_{si}\cos\alpha_{si} + M\sum_{i=1}^{n}\sin^2\alpha_{si} = \sum_{i=1}^{n}\Delta\varepsilon_i\sin\alpha_{si}$$

$$\Delta\varepsilon_o\sum_{i=1}^{n}\cos\alpha_{si} + N\sum_{i=1}^{n}\cos^2\alpha_{si} + M\sum_{i=1}^{n}\sin\alpha_{si}\cos\alpha_{si} = \sum_{i=1}^{n}\Delta\varepsilon_i\cos\alpha_{si}$$

$\Delta\varepsilon_o$, M and N are obtained using the following matrices:

$$\begin{pmatrix} n & \sum_{i=1}^{n}\sin\alpha_{si} & \sum_{i=1}^{n}\cos\alpha_{si} \\ \sum_{i=1}^{n}\sin\alpha_{si} & \sum_{i=1}^{n}\sin^2\alpha_{si} & \sum_{i=1}^{n}\sin\alpha_{si}\cos\alpha_{si} \\ \sum_{i=1}^{n}\cos\alpha_{si} & \sum_{i=1}^{n}\sin\alpha_{si}\cos\alpha_{si} & \sum_{i=1}^{n}\cos^2\alpha_{si} \end{pmatrix} \begin{pmatrix} \Delta\varepsilon_o \\ M \\ N \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n}\Delta\varepsilon_i \\ \sum_{i=1}^{n}\Delta\varepsilon_i\sin\alpha_{si} \\ \sum_{i=1}^{n}\Delta\varepsilon_i\cos\alpha_{si} \end{pmatrix}$$

Then $L = (M^2 + N^2)^{1/2}$

$$\phi_L = \tan^{-1}\left(\frac{M}{N}\right)$$

ϕ_L is the azimuth angle away from true north, at which the maximum mislevel occurs.

$\Delta\varepsilon_o$ = the elevation calibration correction.

Compute the azimuth calibration correction

$$\Delta\alpha_o = \frac{\sum_{i=1}^{n}\Delta\alpha_i}{n}$$

Range Orientation Correction (ΔR_o)

Correct the range as follows:

$$\text{Range error} = \sigma_R \sin\left[\frac{\pi(R_o - \phi_R)}{2048}\right]$$

$$\Delta R = R_s - R_o$$

$$\Delta R_o = \Delta R + \text{Range error}$$

Angular Residuals:

Compute the angular residuals as follows:

$$\delta\epsilon_i = \Delta\epsilon_i - (\Delta\epsilon_o + M \sin \alpha_{si} + N \cos \alpha_{si})$$

$$\delta\alpha_i = \Delta\alpha_i - \Delta\alpha_o$$

Coefficients of Refraction:

The coefficients of refraction are determined by the following formulae:*

$$K_{1e} = (-1.018591636 \times 10^{-3})\ (N)$$

$$K_{2e} = 3.568912557 \times 10^{5} - 4.351144769 \times 10^{3}\ (N) + 2.152067349 \times 10\ (N)^{2} - 4.850656971 \times 10^{-2}\ (N)^{3} + 4.143517896 \times 10^{-5}\ (N)^{4}$$

K_{1R} = According to Table I (N units vs K_{1R})

$$K_{2R} = 1.526309835 \times 10^{4} + 2.851703765 \times 10\ (N) - 3.067090397 \times 10^{-1}\ (N)^{2} + 4.943394167 \times 10^{-4}\ (N)^{3}$$

where N is the index of refraction determined by the contribution due to the pressure and temperature (S units) and the contribution due to the pressure of water vapor (R units)

*Pearson, Kermit E., Kasparek, Dennis D., Tarrant, Lucile N., "The Refraction Correction Developed for the AN/FPS-16 Radar at WSMR" (U), USA SMSA Technical Memorandum 577, November 1958.

Compute:

$$T_a = \frac{6.22 \log^{-1}\left(8.4051 - \frac{2353}{T_w}\right)}{P_o - 10 \log^{-1}\left(8.4051 - \frac{2353}{T_w}\right)}$$

$$T_b = 4 \times 10^{-4} (T_w - T_o)$$

where

T_o = Dry bulb temperature in degrees absolute

T_w = Wet bulb temperature in degrees absolute

P_o = Pressure in millibars

Then the dew point temperature, TD, becomes

$$TD = \frac{2353}{8.4051 - \log\left[\frac{(T_a + T_b)\, P_o}{10\,(T_a + T_b) + 6.22}\right]}$$

$$S(\text{units}) = \frac{77.0\,(P_o)}{T_o}$$

$$R(\text{units}) = \log^{-1}\left[(TD - 273)(.02789) + 1.42618\right]$$

$$N(\text{units}) = R(\text{units}) + S(\text{units})$$

TABLE I

N units vs K_{1R}

N units	K_{1R} (yards)
224 thru 233	-3.03
234 thru 243	-3.04
244 thru 251	-3.05
252 thru 260	-3.06
261 thru 268	-3.07
269 thru 275	-3.08
276 thru 283	-3.09
284 thru 291	-3.10
292 thru 298	-3.11
299 thru 305	-3.12
306 thru 311	-3.13
312 thru 318	-3.14
319 thru 323	-3.15
324 thru 329	-3.16
330 thru 333	-3.17
334 thru 340	-3.18

Data Point Correction:

Having determined the calibration corrections, the observed data at the j^{th} time are corrected by the following formulae:

The data corrected for data shaft eccentricity errors are

$$\epsilon_{1j} = \epsilon_{oj} + e_{\epsilon 1} \sin (\epsilon_{oj} - \phi_{\epsilon 1}) + e_{\epsilon 16} \sin 16 (\alpha_{oj} - \phi_{\epsilon 16})$$

$$\alpha_{1j} = \alpha_{oj} + e_{\alpha 1} \sin (\alpha_{oj} - \phi_{\alpha 1}) + e_{\alpha 16} \sin 16 (\alpha_{oj} - \phi_{\alpha 16})$$

$$R_{1j} = R_{oj} + e_R \sin \left[\frac{\pi (R_{oj} - \phi_R)}{2048} \right]$$

where:

ϵ_{oj} = observed elevation angle at the j^{th} time.

α_{oj} = observed azimuth angle at the j^{th} time.

R_{oj} = observed range at the j^{th} time.

Correct the angles for the error constant (H)

$$\epsilon_{2j} = \sin^{-1} (\sin \epsilon_{1j} \cos |H|)$$

$$\alpha_{2j} = \alpha_{1j} + \sin^{-1} \left(\sin H \left| \frac{\sin \epsilon_{1j}}{\cos \epsilon_{2j}} \right| \right)$$

Correct the elevation angle for the error constant (D), and the azimuth angle for the error constant (C_o)

$$\epsilon_{3j} = \epsilon_{2j} - D \cos \epsilon_{2j}$$

$$\alpha_{3j} = \alpha_{2j} + C_o \sec \epsilon_{2j}$$

Correct the angles for the calibration corrections ($\Delta\epsilon_o$, $\Delta\alpha_o$)

$$\epsilon_{4j} = \epsilon_{3j} + \Delta\epsilon_o$$

$$\alpha_{4j} = \alpha_{3j} + \Delta\alpha_o$$

Correct the range for the range calibration correction (ΔR_o) and beacon delay (Q) by the following equations:

$$R_{2j} = [R_{1j} + \Delta R_o + Q]\ F$$

where

$$F = \frac{\text{free space velocity of propagation}}{\text{Radar velocity of propagation}}$$

The angles are then corrected for the tilt (L) by the following formulae:

$$\varepsilon_{5j} = \sin^{-1}[\sin \varepsilon_{4j} \cos L + \cos \varepsilon_{4j} \sin L \cos(\alpha_{4j} - \phi_L)]$$

$$\alpha_{5j} = \phi_L + \sin^{-1}\left[\frac{\cos \varepsilon_{4j} \sin(\alpha_{4j} - \phi_L)}{\cos \varepsilon_{5j}}\right]$$

$$= \phi_L + \cos^{-1}\left[\frac{-\sin \varepsilon_{4j} + \sin \varepsilon_{5j} \cos L}{\cos \varepsilon_{5j} \sin L}\right]$$

Correcting the angles and range for the refraction correction, the final corrected angles and range become

$$\varepsilon_{6j} = \varepsilon_{5j} - \left|\frac{K_{1e} R_{2j} \cos \varepsilon_{5j}}{K_{2e} + R_{2j} \sin \varepsilon_{5j}}\right|$$

$$\alpha_{6j} = \alpha_{5j}$$

$$R_{6j} = R_{2j} - \left|\frac{K_{1R} R_{2j} \cos \varepsilon_{5j}}{K_{2R} + R_{2j} \sin \varepsilon_{5j}}\right|$$

The direction cosines of the line of sight from the Radar to the missile at time j are determined from the corrected angles by:

$$\lambda_j = \cos \varepsilon_{6j} \cos \alpha_{6j}$$

$$\mu_j = \cos \varepsilon_{6j} \sin \alpha_{6j}$$

$$\nu_j = \sin \varepsilon_{6j}$$

The line of sight is rotated into the WSCS plane by the following rotational matrices.

$$(M)^{-1}\begin{Bmatrix}\lambda_j\\ u_j\\ \upsilon_j\end{Bmatrix} = \begin{Bmatrix}\cos\alpha_j \cos\varepsilon_j\\ \sin\alpha_j \cos\varepsilon_j\\ \sin\varepsilon_j\end{Bmatrix}$$

where

$$(M)^{*} = \begin{Bmatrix}M_{11} & M_{21} & M_{31}\\ M_{12} & M_{22} & M_{32}\\ M_{13} & M_{23} & M_{33}\end{Bmatrix}$$

and

$$M_{11} = \sin\phi_0 \sin\phi_k \cos(\lambda_0 - \lambda_k) + \cos\phi_0 \cos\phi_k$$

$$M_{12} = -\sin\phi_k \sin(\lambda_0 - \lambda_k)$$

$$M_{13} = -\cos\phi_0 \sin\phi_k \cos(\lambda_0 - \lambda_k) + \sin\phi_0 \cos\phi_k$$

$$M_{21} = \sin\phi_0 \sin(\lambda_0 - \lambda_k)$$

$$M_{22} = \cos(\lambda_0 - \lambda_k)$$

$$M_{23} = -\cos\phi_0 \sin(\lambda_0 - \lambda_k)$$

$$M_{31} = -\sin\phi_0 \cos\phi_k \cos(\lambda_0 - \lambda_k) + \sin\phi_k \cos\phi_0$$

$$M_{32} = \cos\phi_k \sin(\lambda_0 - \lambda_k)$$

$$M_{33} = \cos\phi_0 \cos\phi_k \cos(\lambda_0 - \lambda_k) + \sin\phi_0 \sin\phi_k$$

*The derivation of this matrix is described in ADDENDUM I.

ϕ_0, λ_0 = Geodetic Positions of the WSCS Point

ϕ_k, λ_k = Geodetic Positions of the k^{th} Radar

The direction cosines are then converted to rectangular coordinates in the WSCS plane as follows:

$$x_j'' = C_R\ R6_j(\cos\varepsilon_j\ \cos\alpha_j)$$

$$y_j'' = C_R\ R6_j(\cos\varepsilon_j\ \sin\alpha_j)$$

$$z_j'' = C_R\ R6_j\ \sin\varepsilon_j$$

C_R = conversion factor from yards to desired units. The coordinates are then translated to the desired origin by the following equations:

$$x_j' = x_j'' + \Delta x$$

$$y_j' = y_j'' + \Delta y$$

$$z_j' = z_j'' + \Delta z$$

where Δs, Δy, Δz are the WSCS coordinates of the radar with respect to the desired origin.

The position data may then be rotated into the tangent plane of the desired origin by the following rotational matrices

$$(A_z)\ (M_L) \begin{pmatrix} x_j^1 \\ y_j^1 \\ z_j^1 \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix}$$

where M_L is the M matrix as defined previously, except that ϕ_L, λ_L replaces ϕ_k, λ_k.

ϕ_L, λ_L, = Geodetic Positions of the desired origin (usually the Launcher)

and A_z is the rotational matrix that orients the system to the line of fire.

$$(A_z) = \begin{Bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

where A is the azimuth of fire from true north positive clockwise.

The range, azimuth and elevation may be computed with respect to the launcher by the following equations

$$R_{Lj} = \frac{(x_j^2 + y_j^2 + z_j^2)^{1/2}}{C_R}$$

$$\alpha_{Lj} = \sin^{-1} \frac{y_j}{(x_j^2 + y_j^2)^{1/2}}$$

$$\varepsilon_{Lj} = \sin^{-1} \frac{z_j}{(x_j^2 + y_j^2 + z_j^2)^{1/2}}$$

AN/FPS-16 DATA CORRECTION SHEET

INFORMATION

INSTRUMENTATION RADAR NO.____________________ DATE__________

TARGET____________ ROUND NO.________ OPERATION______ TIME______

AZIMUTH OCTAL__________________ DATA SAMPLING RATE__________

REMARKS:

IN AUTO

CORRECTION DATA

WET BULB____________ DRY BULB______________ PRESSURE________

SKIN GATE OCTAL__________________ BEACON DELAY____________

DEVIATIONS FROM VERTICAL CROSS HAIR____________________

A_1__________ A_2__________ E_1__________ E_2__________

TARGET OBSERVATIONS (OCTAL)

TARGET NO	1	2	3	4	5	6
AZIMUTH						
ELEVATION						

FIGURE 2

DERIVATION OF DEW-POINT TEMPERATURE

The dew-point temperature, T_D, is defined as the temperature to which moist air must be cooled, while keeping both pressure and mixing ratio constant, until the moist air just reaches saturation with respect to the water. The dew-point temperature can be expressed as a function of the partial vapor pressure of humid air, e_p, which in turn is found from the wet bulb temperature, dry bulb temperature and mixing ratios.

The basic relationship used to express the dew-point temperature in terms of partial vapor pressure is derived from the Clausius-Clapeyron equation, (sometimes called simply the Clapeyron equation). Whenever a substance changes phase (melts, freezes, evaporates or condenses) a quantity of heat must be supplied or taken away from the substance while the temperature remains constant. This quantity is called the latent heat (L) of the phase change. If, for example, as in the computation of dew-point temperatures, the two phases are water and vapor respectively, then L is the latent heat of evaporation. The Clausius-Clapeyron equation relates this latent heat to the discontinuous change in volume accompanying a phase change and to the slope of the curve of saturation vapor pressure vs. temperature:

$$L = (v_2 - v_1)\, T \, \frac{de_s}{dT} \tag{1}$$

where v_2, v_1 = specific volume of vapor and water respectively

e_s = saturation vapor pressure

T = temperature.

When the specific volume of vapor is much greater than that of water, (i.e., $v_2 >> v_1$), and the vapor is assumed to obey the perfect gas law $PV = RT$, R being the gas constant for vapor, equation (1) can be written in the form:

$$\frac{de_s}{e_s} = \frac{L\, dT}{R\, T^2} \tag{2}$$

Assuming that the latent heat is a function of temperature, $L = b - cT$, equation (2) can be integrated to give

$$\ln e_s = -\frac{b}{RT} - \frac{c}{R} \ln (T) + \text{Constant} \tag{3}$$

or, letting

$$B = \frac{b}{R},$$

$$C = \frac{c}{R},$$

A = the constant of integration,

$$\ln e_s = \left(-\frac{B}{T} - C \ln (T) + A\right) \tag{4}$$

Substituting numerical values for the constants, and measuring the temperature in °K, the value of the saturation vapor pressure, e_s, in centibars is found from

$$\ln e_s = \left(\frac{-6763.61}{T} - 4.9283 \ln (T) + 51.9274\right) \tag{5}$$

Converting to common logarithms equation (5) becomes:

$$\log e_s = \left(\frac{-2937.4}{T} - 4.9283 \log (T) + 22.5518\right) \tag{6}$$

or equivalently:

$$\log e_s = \frac{T[22.5518 - 4.9283 \log (T)] - 2937.4}{T} \tag{7}$$

An expression for log (T) is found, assuming the mean of the expected temperature range to be 280°K.

$$\log (T) = \log \left(\frac{T}{280}\right) + \log (280) \tag{8}$$

$$= \log e \ln \left(\frac{T}{280}\right) + \log (280)$$

$$= .43429 \ln \left(\frac{T}{280}\right) + \log (280)$$

A series expansion for $\ln\left(\frac{T}{280}\right)$ when $\left(\frac{T}{280}\right) > \frac{1}{2}$, neglecting higher order terms which are insignificant, yields

$$\ln\left(\frac{T}{280}\right) = \left(\frac{\frac{T}{280} - 1}{\frac{T}{280}}\right) = \left(1 - \frac{280}{T}\right) \quad (9)$$

Substituting equation (9) in equation (8) yields

$$\log (T) = .43429 \left(1 - \frac{280}{T}\right) + 2.44716731 \quad (10)$$

Using this expression in equation (7) gives the equation for saturation vapor pressure as a function of temperature:

$$\log e_s = \frac{T\left\{22.5518 - 4.9283 \left[.43429\left(1 - \frac{280}{T}\right) + 2.44716731\right]\right\} - 2937.4}{T}$$

$$= \frac{T\left[22.5518 - 4.9283 (2.44716731) - 2.0910284\left(1 - \frac{280}{T}\right)\right] - 2937.4}{T}$$

$$= \frac{T (8.4051) - 2353}{T} \quad (11)$$

From this equation it can be seen that the dew-point temperature, T_D, is related to the partial vapor pressure of humid air, e_p, in centibars, by:

$$T_D = \frac{2353}{8.4051 - \log e_p} \quad (12)$$

The partial vapor pressure is found from the wet bulb temperature, dry bulb temperature and mixing ratios. The wet-bulb temperature is defined to be the temperature to which air may be cooled by evaporating water into it at constant pressure until saturated. The latent heat of evaporation is thought of as coming from the air. The mixing ratio, that is, the ratio of the mass of water vapor present to the mass of dry air containing the vapor, is not kept constant.

If a stream of air at a certain pressure, temperature and mixing ratio flows past a thermometer bulb which is covered with a damp cloth, water will be evaporated from the cloth by the flowing air. The thermometer bulb will be cooled by the evaporation. When an equilibrium condition is reached, that is, when the loss of heat by air flowing past the wet bulb equals the sensible heat which is transformed to latent heat, the following energy equation holds:

$$(T_o - T_w)(cp + wc_p') = (w' - w)\,Lw \tag{13}$$

where T_o = dry bulb temperature (temperature of approaching air)

T_w = wet bulb temperature (temperature of leaving air)

w = mixing ratio of approaching air

w' = mixing ratio of leaving air (saturated mixing ratio)

c_p = specific heat at constant pressure of dry air

cp' = specific heat at constant pressure of water vapor

L_w = latent heat of vaporization at wet bulb temperature

T_o and T_w are in degrees absolute.

The mixing ratio, w, is found from

$$w = \frac{\rho_w}{\rho_d} \tag{14}$$

where $\rho_w = \dfrac{m_w\, 10\, ep}{RT}$ = density of water vapor

$\rho_d = \dfrac{m_d\,(P - 10\, ep)}{RT}$ = density of dry air

and e_p is in centibars and P is in millibars.

m_d = apparent molecular weight of dry air = 28.966

m_w = apparent molecular weight of water = 18.0159

m_w/m_d = 0.62197

then

$$w = \frac{6.22\, e_p}{P - 10\, e_p} \tag{15}$$

The value of the saturation mixing ratio is defined to be the maximum value w may have at a specified temperature and pressure:

$$w' = \frac{6.22\ e_s}{P - 10\ e_s} \tag{16}$$

where e_s = saturation vapor pressure in centibars

P = pressure in millibars.

Solving equation (13) for w,

$$w = \frac{w' - (T_o - T_w)\dfrac{c_p}{L_w}}{\dfrac{c_p'}{L_w}(T_o - T_w) + 1} \tag{17}$$

Equating equations (15) and (17) and solving for e_p yields

$$e_p = \left(\frac{w}{6.22 + 10w}\right) P = \frac{\left[w' - (T_o - T_w)\dfrac{c_p}{L_w}\right] P}{10\left[w' - (T_o - T_w)\left(\dfrac{c_p}{L_w} - .622\dfrac{c_p'}{L_w}\right)\right] + 6.22} \tag{18}$$

where

c_p = 1.003 joules/gram/°K

c_p' = 1.81 joules/gram/°K

L_w = $(2502 - 2.38\ T_w)$, but since L_w varies only slightly with the temperature, $L \cong 2502$ joules/gram.

Substituting these values in equation (18) yields the solution

$$e_p \approx \frac{[w' + (T_o - T_w)(-4 \times 10^{-4})]\ P}{10\ [w' + (T_o - T_w)(4.9 \times 10^{-5})] + 6.22} \tag{19}$$

A very close approximation to this may be found by assuming in equation (17) that

$$\left(\frac{cp'}{l_w}(T_o - T_w) + 1\right) \approx 1.$$

Then

$$e_p = \frac{[w' + (T_o - T_w)(-4 \times 10^{-4})]\ P}{10\ [w' + (T_o - T_w)(-4 \times 10^{-4})] + 6.22} \tag{20}$$

which is used in the reduction of the dew-point temperature.

From equations (11) and (16) it is easily seen that

$$w' = \frac{6.22 \log^{-1}\left[8.4051 - \frac{2353}{T_w}\right]}{P - 10 \log^{-1}\left[8.4051 - \frac{2353}{T_w}\right]} \tag{21}$$

If $T_a = w'$

$T_b = (-4 \times 10^{-4})(T_o - T_w) = 4 \times 10^{-4}\ (T_w - T_o)$

then equation (18) becomes

$$e_p = \frac{(T_a + T_b)\ P}{10(T_a + T_b) + 6.22} \tag{22}$$

where e_p is in centibars.

Substituting equation (22) into equation (12), the dew-point temperature becomes

$$T_D = \frac{2353}{8.4051 - \log\left[\frac{(T_a + T_b)\ P}{10(T_a + T_b) + 6.22}\right]}$$

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B. POSITION DATA

III Launcher Data

LAUNCHER DATA

Introduction

Launcher data provide the distance a missile moves along its launcher, and the velocity and acceleration of the missile along the launcher. The data are obtained from a high speed fixed camera. A fixed camera may be defined as a camera that utilizes a fixed field of view for photographing missile flights. The fixed cameras used to obtain these data at WSMR may be high speed 70mm or high speed Mitchell cameras.

The procedure involves finding the distance the missile travels along the launcher when given the distance the missile image moves along the image of the launcher in the film plane.

Film Reading:

The film is read on the Telereadex (which is described in the cinetheodolite section).

To aid in simplifying the reduction the camera is tilted through an angle equal to the elevation angle of the launcher. The top edge of the launcher image is then parallel to the lower edge of the film frame. The contractor places tapes of a certain width along the launcher at fixed intervals. The edge of one of these tapes is chosen as an origin for the film measurement.

The film is read in the following manner:

a. The telereadex is zeroed on the first visible tape.

b. Machine counts are read from the zero tape to three other tapes.

c. A reference point is chosen on the missile. Starting two frames before blast, and without re-zeroing, this reference point is read and the machine counts recorded for each frame until the missile leaves the launcher.

Standard IRIG timing (Format A) is used on the film.

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Mathematical Derivation:

In the following derivation, reference is made to the figure below:

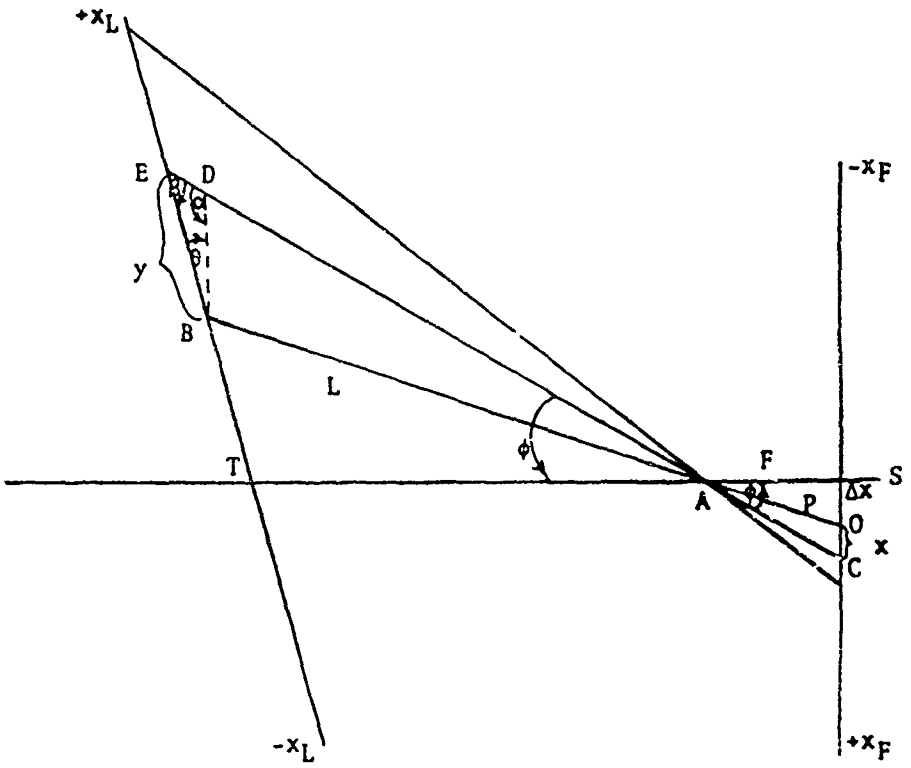

FIGURE 1

Line x_F is the x-axis of the film plane

Line x_L is the x-axis of the launcher plane

ST is the optical axis of the camera and is perpendicular to x_F

A is the focal point of the camera

y is the unknown distance on the launcher

x is the projection of y on the film

BD is parallel to x_F by construction

θ is the angle between the launcher and the film plane

$\alpha = 90° + \phi$

$\beta = 90° - (\theta + \phi)$

Since

$$\Delta ABD \sim \Delta AOC$$

then $$\frac{BD}{x} = \frac{L}{P}$$

$$BD = \frac{L\,x}{P} \qquad (1)$$

By applying the law of sines to ΔBDE we find

$$\frac{y}{\sin\alpha} = \frac{BD}{\sin\beta}$$

$$\frac{y}{\sin(90^\circ + \phi)} = \frac{BD}{\sin[90^\circ - (\theta + \phi)]}$$

$$\frac{y}{\cos\phi} = \frac{BD}{\cos(\theta + \phi)}$$

$$y = \frac{BD\cos\phi}{\cos(\theta + \phi)}$$

$$y = \frac{BD\cos\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} \qquad (2)$$

By substituting equation 1 in equation 2

$$y = \frac{\frac{L}{P}\,x\cos\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

$$y = \frac{x}{\frac{P}{L}\cos\theta - \frac{P}{L}\tan\phi\sin\theta}$$

From the right ΔASC in Fig. 1 it is seen that

$$\tan\phi = \frac{x + \Delta x}{F}$$

by substitution

$$y = \frac{x}{\frac{P}{L}\cos\theta - \frac{P}{L}\left(\frac{x + \Delta x}{F}\right)\sin\theta}$$

$$= \frac{x}{\frac{P}{L}\cos\theta - \frac{P\,\Delta x}{LF}\sin\theta - \frac{Px}{LF}\sin\theta}$$

$$= \frac{x}{\frac{P}{L}(\cos\theta - \frac{\Delta x}{F}\sin\theta) - \frac{Px}{LF}\sin\theta} \quad (4)$$

If

$$a = \frac{P}{L}(\cos\theta - \frac{\Delta x}{F}\sin\theta)$$

$$b = \frac{-P}{LF} \quad \sin\theta$$

where a and b are constants,

then equation (4) becomes:

$$y = \frac{x}{a + bx} \quad (5)$$

Since the distance (y) between the tapes on the launcher is known and the corresponding values of x are read on the film, the values of x and y may be substituted in equation (5) for two distances thus obtaining two equations. a and b may be solved for in the following manner:

If y_i is the known distance from the origin on the launcher to the i^{th} tape and x_i is the corresponding measured distance on the film, then

$$y_1 = \frac{x_1}{a + b\,x_1}$$

$$y_2 = \frac{x_2}{a + b\,x_2}$$

$$a\,y_1 + b\,x_1\,y_1 = x_1$$

$$a\,y_2 + b\,x_2\,y_2 = x_2$$

and solving by determinants

$$a = \frac{\begin{vmatrix} x_1 & x_1\,y_1 \\ x_2 & x_2\,y_2 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1\,y_1 \\ y_2 & x_2\,y_2 \end{vmatrix}} = \frac{x_1\,x_2\,y_2 - x_1\,x_2\,y_1}{x_2\,y_1\,y_2 - x_1\,y_1\,y_2}$$

$$b = \frac{\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1\,y_1 \\ y_2 & x_2\,y_2 \end{vmatrix}} = \frac{y_1\,x_2 - x_1\,y_2}{x_2\,y_1\,y_2 - x_1\,y_1\,y_2}$$

These values may be substituted in

$$y_3 = \frac{x_3}{a + b\,x_3}$$

as a check on the computation.

The a and b are then substituted in the equation

$$y_t = \frac{x_t}{a + b\,x_t}$$

where y_t is the distance the missile has moved at time (t) and x_t is the corresponding value of y_t on the film.

Velocity and acceleration may then be computed as described in the Velocity and Acceleration section.

LAUNCHER DATA REFERENCES

Dale, Richard H. SPECIAL REPORT ON LAUNCHER TRAJECTORY DATA
SRB, Data Reduction Division, IRM, White Sands Missile Range, New Mexico, 13 June 1960

C. ATTITUDE DATA

C. ATTITUDE DATA

I Telescope Orientation System for Mislevel

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TELESCOPE ORIENTATION SYSTEM FOR MISLEVEL CORRECTION

Four target boards have been placed an equal distance from and approximately every 90° about each telescope camera. Each target board contains two diamond shaped targets (Fig. 1) and has been set up so that a line through the center of the two diamonds, which we will call the target axis, is precisely parallel to the local tangent plane of the camera.

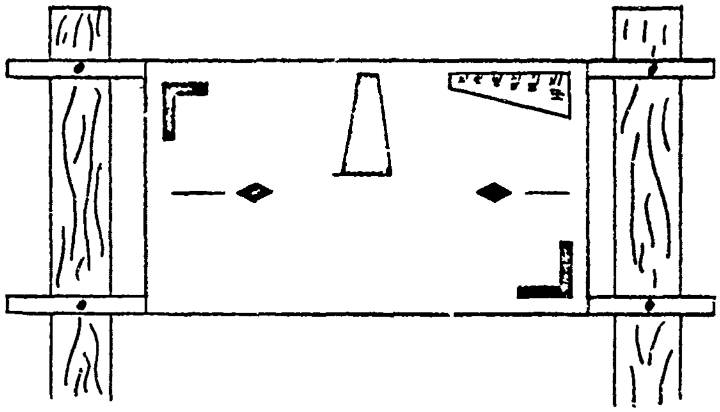

FIGURE 1

Immediately before each missile firing having a requirement for attitude data, orientation shots are taken for each camera. This is done by sighting the telescope on each of the four target boards in turn and photographing them. The data from these orientation shots from each telescope are used to determine the mislevel of that particular camera, the phase angle of this mislevel, and the frame edge referencing correction. These constants and the position data of the missile are then used to find the correction to be applied to the V-angle reading of the missile. The calculation of this mislevel correction also requires the surveyed azimuth and elevation angles from the camera to the target board in the WSTM system, the launcher and camera coordinates in the WSCS system and the azimuth of fire to which the position data are referenced.

The coordinate system used in the reduction of the missile position data is defined as the XYZ system (launcher tangent plane). In deriving the equations for this orientation system it is necessary to set up a new coordinate system, X'Y'Z'. The X' axis is positive along the line of sight from the camera to the target board. The Y' axis lies in the film plane perpendicular to the X' axis and is directed positive to the right and parallel to the XY plane. The Z' axis also lies in the film plane perpendicular to the X'Y' plane, and positive up. (Fig. 2).

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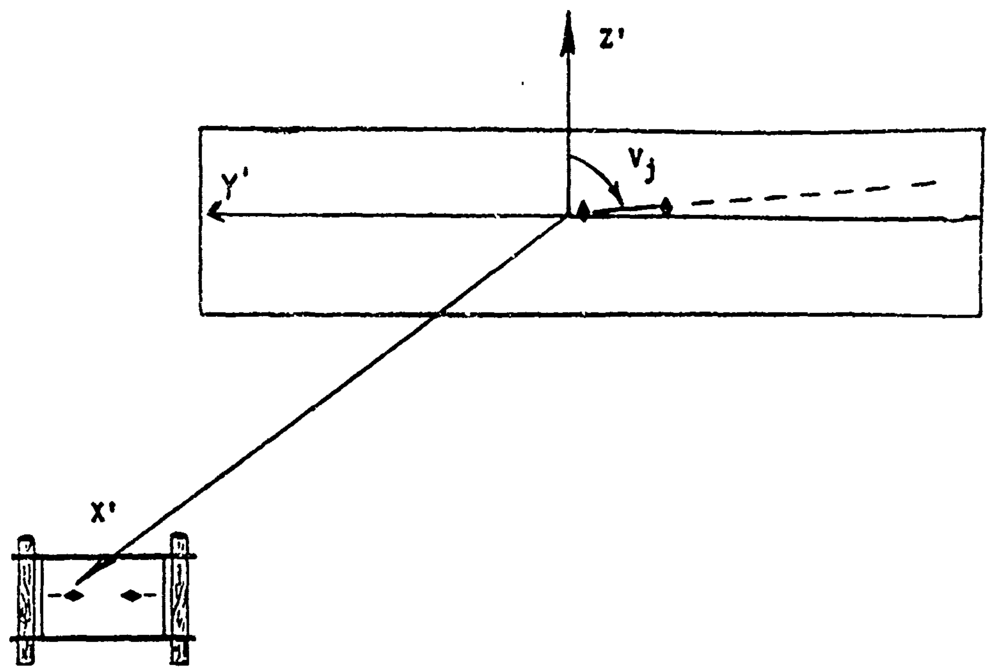

FIGURE 2

The V-angle is measured clockwise from the Z' axis to the target axis image.

We will first consider the target axis. If the surveyed azimuth and elevation angles from the camera to the jth target are αs_j and εs_j then the azimuth and elevation angles of the jth target axis in the local tangent plane of the camera are:

$$\alpha_{t_j} = \alpha s_j + 90°$$

$$\varepsilon_{t_j} = 0$$

The direction cosines of the jth target axis in the local tangent plane are:

$$a_j = \cos \alpha_{t_j}$$

$$b_j = \sin \alpha_{t_j}$$

$$c_j = 0$$

These direction cosines are then rotated into the launcher tangent plane by the following rotational matrices:

$$\begin{pmatrix} a_{L_j} \\ b_{L_j} \\ c_{L_j} \end{pmatrix} = [A]\ [R] \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} \qquad (1)$$

where

$$[A] = \begin{bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

and A is the azimuth of fire to which the position data are referenced.

$$[R] = \begin{bmatrix} \cos \alpha_k & \sin \alpha_k & 0 \\ -\sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \varepsilon_k & 0 & -\sin \varepsilon_k \\ 0 & 1 & 0 \\ \sin \varepsilon_k & 0 & \cos \varepsilon_k \end{bmatrix} \begin{bmatrix} \cos \alpha_k & -\sin \alpha_k & 0 \\ \sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2 \alpha_k + \cos^2 \alpha_k \cos \varepsilon_k & \sin \alpha_k \cos \alpha_k (\cos \varepsilon_k - 1) & \cos \alpha_k \sin \varepsilon_k \\ \sin \alpha_k \cos \alpha_k (\cos \varepsilon_k - 1) & \cos^2 \alpha_k + \sin^2 \alpha_k \cos \varepsilon_k & \sin \alpha_k \sin \varepsilon_k \\ -\cos \alpha_k \sin \varepsilon_k & -\sin \alpha_k \sin \varepsilon_k & \cos \varepsilon_k \end{bmatrix} \tag{3}$$

where α_k and ε_k are the angles thru which the direction cosines of the target axis are rotated to obtain the direction cosines in the launcher tangent plane. The angles are:

$$\alpha_k = \tan^{-1}\left(\frac{Y_k - Y_L}{X_k - X_L}\right) \tag{4}$$

$$\varepsilon_k = \tan^{-1} \frac{\left[(X_k - X_L)^2 + (Y_k - Y_L)^2\right]^{1/2}}{\rho + Z_k - Z_L} \tag{5}$$

where X_k, Y_k, Z_k = WSCS coordinates of the k^{th} camera

X_L, Y_L, Z_L = WSCS coordinates of the Launcher

ρ = radius of curvature from the launcher to the k^{th} camera

The radius of curvature, ρ, is found from the following equation:

$$\rho = \frac{RN}{R \sin^2 \alpha_k + N \cos^2 \alpha_k} = \frac{R}{\left(\frac{R}{N} - 1\right) \sin^2 \alpha_k + 1} \tag{6}$$

where

R = radius of curvature in the WSCS origin meridian = 20,847,227.51 ft.

N = radius of curvature in the prime vertical = 20,946,965.81 ft.

It is now necessary to obtain the surveyed azimuth and elevation angles of the j^{th} target board in the launcher tangent plane.

The direction cosines of the surveyed azimuth and elevation angles from the camera to the j^{th} target board in the local tangent plane are:

$$a_{sj} = \cos \alpha_{sj} \cos \varepsilon_{sj}$$

$$b_{sj} = \sin \alpha_{sj} \cos \varepsilon_{sj}$$

$$c_{sj} = \sin \varepsilon_{sj}$$

These direction cosines are rotated into the launcher tangent plane by the following rotational matrices:

$$\begin{pmatrix} a'_{Lj} \\ b'_{Lj} \\ c'_{Lj} \end{pmatrix} = [A] \; [R] \begin{pmatrix} a_{sj} \\ b_{sj} \\ c_{sj} \end{pmatrix} \tag{7}$$

where the rotational matrices [A] and [R] are as previously defined in equations (2) and (3).

The surveyed azimuth and elevation angles in the launcher tangent plane are:

$$\alpha'_{sj} = \tan^{-1} \left(\frac{b'_{Lj}}{a'_{Lj}} \right) \tag{8}$$

$$\varepsilon'_{sj} = \tan^{-1} \left[\frac{c'_{Lj}}{(a'^2_{Lj} + b'^2_{Lj})^{1/2}} \right] \tag{9}$$

Now that we have the direction cosines of the target axis and the surveyed angles from the camera to the j^{th} target board in the launcher tangent plane (X, Y, Z system) we will rotate the direction cosines of the j^{th} target axis into the X'Y'Z' system by the following rotational matrix:

$$\begin{bmatrix} a'_j \\ b'_j \\ c'_j \end{bmatrix} = \begin{bmatrix} \cos\alpha'_{sj}\cos\varepsilon'_{sj} & \sin\alpha'_{sj}\cos\varepsilon'_{sj} & \sin\varepsilon'_{sj} \\ -\sin\alpha'_{sj} & \cos\alpha'_{sj} & 0 \\ -\cos\alpha'_{sj}\sin\varepsilon'_{sj} & -\sin\alpha'_{sj}\sin\varepsilon'_{sj} & \cos\varepsilon'_{sj} \end{bmatrix} \begin{bmatrix} a_{Lj} \\ b_{Lj} \\ c_{Lj} \end{bmatrix} \tag{10}$$

The true V-angle of the j^{th} target axis in the X'Y'Z' system is:

$$V_{tj} = \tan^{-1}\left(\frac{b'_j}{c'_j}\right) \tag{11}$$

The telescope may not be level, that is, the vertical axis about which the camera rotates may not be parallel to the Z axis of the reference system. Thus the Y' axis would not remain parallel to the XY plane. This results in a V-angle error which will vary as the camera rotates in azimuth and may be expressed in terms of L, the angle of maximum tilt or mislevel, and ϕ_L, the phase angle in the azimuth plane at maximum tilt. Also, the V-angle reading reference axis (frame edge) may not be perpendicular to the Y' axis. This error may be expressed as ΔV_0. These errors are determined in the following manner:

Assume that the error equation is of the form:

$$\Delta V_j = (V_{tj} - V_j^{\circ}) = \Delta V_0 + L\cos(\alpha'_{sj} - \phi_L) \tag{12}$$

L = maximum tilt

ϕ_L = phase angle of maximum tilt

α'_{sj} = azimuth angle from the camera to the j^{th} target board in the launcher tangent plane

V_{tj} = true V-angle of the j^{th} target axis

V_j° = observed V-angle of the j^{th} target axis

Equation (12) may be written as:

$$\Delta V_j = \Delta V_0 + M\sin\alpha'_{sj} + N\cos\alpha'_{sj}$$

where $M = L\sin\phi_L$

$N = L\cos\phi_L$

Using the least squares solution the following matrices are formed:

$$\begin{bmatrix} n & \Sigma \sin \alpha'_{sj} & \Sigma \cos \alpha'_{sj} \\ \Sigma \sin \alpha'_{sj} & \Sigma \sin^2 \alpha'_{sj} & \Sigma \sin \alpha'_{sj} \cos \alpha'_{sj} \\ \Sigma \cos \alpha'_{sj} & \Sigma \sin \alpha'_{sj} \cos \alpha'_{sj} & \Sigma \cos^2 \alpha'_{sj} \end{bmatrix} \begin{bmatrix} \Delta V_0 \\ V \\ N \end{bmatrix} = \begin{bmatrix} \Sigma \Delta V_j \\ \Sigma \Delta V_j \sin \alpha'_{sj} \\ \Sigma \Delta V_j \cos \alpha'_{sj} \end{bmatrix}$$

where n = number of targets used.

ΔV_0, M and N are then solved for. These values are substituted in the following equation to obtain the correction (ΔV_i) to be applied to the i^{th} V-angle reading of the telescope:

$$\Delta V_i = \Delta V_0 + M \sin \alpha_i + N \cos \alpha_i$$

where

$$\alpha_i = \tan^{-1} \left(\frac{Y_m - Y_c}{X_m - X_c} \right)$$

X_m, Y_m = coordinates of the missile at the i^{th} time.

X_c, Y_c = launcher tangent plane coordinates of the k^{th} telescope.

Then the corrected V-angle is given by

$$V_i = V_i^{*} + \Delta V_i$$

The V_i's are then used as input data for the N-station attitude reduction.

Computational Order of Formulae

Compute the rotational matrix

$$[A] = \begin{bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where A is the azimuth of fire to which the position data of the missile are referenced.

Compute for the k^{th} camera:

1. $$\alpha_k = \tan^{-1}\left(\frac{Y_k - Y_L}{X_k - X_L}\right)$$

where

X_k, Y_k, Z_k = WSCS coordinates of the K^{th} camera

X_L, Y_L, Z_L = WSCS coordinates of the Launcher

2. The radius of curvature, ρ, from the launcher to the k^{th} camera

$$\rho = \frac{R}{\left(\frac{R}{N} - 1\right)\sin^2 \alpha_k + 1}$$

where

R = 20,847,227.51 ft.

N = 20,946,963.81 ft.

3. $$\varepsilon_k = \tan^{-1} \frac{[(X_k - X_L)^2 + (Y_k - Y_L)^2]^{\frac{1}{2}}}{\rho + Z_k - Z_L}$$

4. $$[R] = \begin{bmatrix} \sin^2 \alpha_k + \cos^2 \alpha_k \cos \varepsilon_k & \sin \alpha_k \cos \alpha_k (\cos \varepsilon_k - 1) & \cos \alpha_k \sin \varepsilon_k \\ \sin \alpha_k \cos \alpha_k (\cos \varepsilon_k - 1) & \cos^2 \alpha_k + \sin^2 \alpha_k \cos \varepsilon_k & \sin \alpha_k \sin \varepsilon_k \\ -\cos \alpha_k \sin \varepsilon_k & -\sin \alpha_k \sin \varepsilon_k & \cos \varepsilon_k \end{bmatrix}$$

Compute for each target about the k^{th} camera:

1. Azimuth and elevation angles of the j^{th} target axis in the local tangent plane.

$$\alpha_{t_j} = \alpha_{s_j} + 90°$$

$$\varepsilon_{t_j} = 0$$

where α_{s_j} is the surveyed azimuth angle from the camera to the j^{th} target board.

2. Direction cosines of the j^{th} target axis in the local tangent plane

$$a_j = \cos \alpha_{t_j}$$

$$b_j = \sin \alpha_{t_j}$$

$$c_j = 0$$

3. Direction cosines of the j^{th} target axis in the launcher tangent plane

$$\begin{pmatrix} a_{L_i} \\ b_{L_i} \\ c_{L_i} \end{pmatrix} = [A]\ [R] \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix}$$

4. Direction cosines of the surveyed azimuth and elevation angles from the camera to the j^{th} target board in the local tangent plane.

$$a_{s_j} = \cos \alpha_{s_j} \cos \varepsilon_{s_j}$$

$$b_{s_j} = \sin \alpha_{s_j} \cos \varepsilon_{s_j}$$

$$c_{s_j} = \sin \varepsilon_{s_j}$$

5. Direction cosines of the surveyed angles in the launcher tangent plane.

$$\begin{pmatrix} a'_{L_j} \\ b'_{L_j} \\ c'_{L_j} \end{pmatrix} = [A]\ [R] \begin{pmatrix} a_{s_j} \\ b_{s_j} \\ c_{s_j} \end{pmatrix}$$

6. Surveyed angles in the launcher tangent plane.

$$\alpha'_{sj} = \tan^{-1}\left(\frac{b'_{Lj}}{a'_{Lj}}\right)$$

$$\epsilon'_{sj} = \tan^{-1}\left[\frac{c'_{Lj}}{\left(a'^{2}_{Lj} + b'^{2}_{Lj}\right)^{1/2}}\right]$$

7. Direction cosines of the j^{th} target axis in the X'Y'Z' system.

$$\begin{bmatrix} a'_j \\ b'_j \\ c'_j \end{bmatrix} = \begin{bmatrix} \cos\alpha'_{sj}\cos\epsilon'_{sj} & \sin\alpha'_{sj}\cos\epsilon'_{sj} & \sin\epsilon'_{sj} \\ -\sin\alpha'_{sj} & \cos\alpha'_{sj} & 0 \\ -\cos\alpha'_{sj}\sin\epsilon'_{sj} & -\sin\alpha'_{sj}\sin\epsilon'_{sj} & \cos\epsilon'_{sj} \end{bmatrix} \begin{bmatrix} a_{Lj} \\ b_{Lj} \\ c_{Lj} \end{bmatrix}$$

8. True V-angle for the j^{th} target axis.

$$V_{tj} = \tan^{-1}\left(\frac{b'_j}{c'_j}\right)$$

9. $\Delta V_j = V_{tj} - V^{o}_j$

where

V^{o}_j = observed V-angle of the j^{th} target axis.

For the k^{th} camera compute:

1. ΔV_o, M and N are from the following:

$$\begin{bmatrix} n & \Sigma\sin\alpha'_{sj} & \Sigma\cos\alpha'_{sj} \\ \Sigma\sin\alpha'_{sj} & \Sigma\sin^2\alpha'_{sj} & \Sigma\sin\alpha'_{sj}\cos\alpha'_{sj} \\ \Sigma\cos\alpha'_{sj} & \Sigma\sin\alpha'_{sj}\cos\alpha'_{sj} & \Sigma\cos^2\alpha'_{sj} \end{bmatrix} \begin{bmatrix} \Delta V_o \\ M \\ N \end{bmatrix} = \begin{bmatrix} \Sigma\Delta V_j \\ \Sigma\Delta V_j\sin\alpha'_{sj} \\ \Sigma\Delta V_j\cos\alpha'_{sj} \end{bmatrix}$$

n = number of targets used.

For the i^{th} position data of the missile compute:

1. Azimuth of the k^{th} camera with respect to the i^{th} missile position

$$\alpha_i = \tan^{-1}\left(\frac{Y_m - Y_c}{X_m - X_c}\right)$$

where X_m, Y_m = coordinates of the missile at the i^{th} time

X_C, Y_C = launcher tangent plane coordinates of the k^{th} camera.

2. The V-angle correction (ΔV_i) to be applied to the i^{th} observed V-angle

$$\Delta V_i = \Delta V_0 + M \sin \alpha_i + N \cos \alpha_i$$

3. The corrected V-angle at the i^{th} time

$$V_i = V_i^{\circ} + \Delta V_i$$

where V_i° = observed V-angle reading at the i^{th} time.

C. ATTITUDE DATA

II N-Station Attitude Reduction

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N-Station Attitude Solution

In this N-Station Attitude Solution a least squares procedure is employed twice. In the first application the residuals are functions of the V-angles. The attitude resulting from the first least squares procedure is then further refined by minimizing the sum of the squares of the errors in the V-angles.

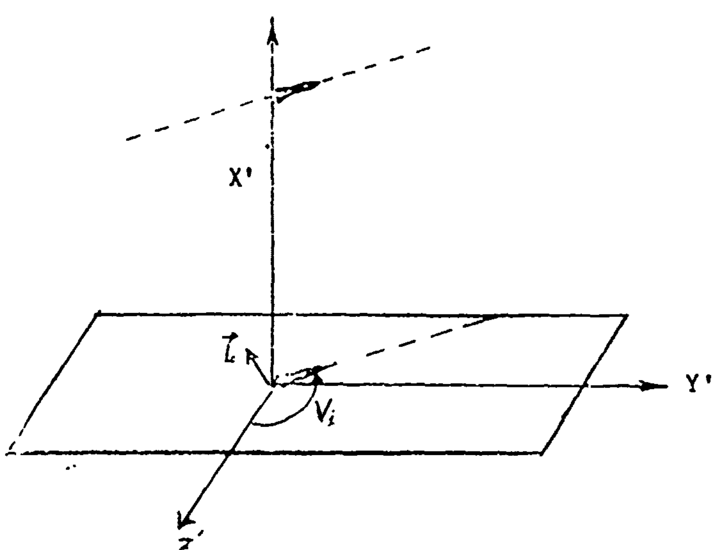
The coordinate system used in the reduction of the position data is defined as the XYZ-system. In deriving the equations it is necessary to set up a new coordinate system X', Y', Z'. The X' axis is positive along the line of sight from the camera to the missile. The Y' axis lies in the film plane perpendicular to the X' axis and directed positive to the right and parallel to the XY plane. The Z' axis also lies in the film plane and is perpendicular to the X'Y' plane and positive up.

In this coordinate system the direction cosines of the vector lying along the missile image are sine and cosine functions of the V-angle. The V-angle is measured clockwise from the Z' axis to the missile image. These direction cosines may be rotated into the XYZ-system by the following rotational matrices.

$$\begin{pmatrix} l_i \\ m_i \\ n_i \end{pmatrix} = \begin{pmatrix} \cos\alpha_i \cos\varepsilon_i & -\sin\alpha_i & -\cos\alpha_i \sin\varepsilon_i \\ \sin\alpha_i \cos\varepsilon_i & \cos\alpha_i & -\sin\alpha_i \sin\varepsilon_i \\ \sin\varepsilon_i & 0 & \cos\varepsilon_i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \sin V_i \\ \cos V_i \end{pmatrix}$$

l_i, m_i, n_i are the direction cosines of the image vector in the XYZ-system. α_i, ε_i are the azimuth and elevation angles of the missile with respect to the i^{th} station.

The following figure represents the situation that exists at each station:



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$\vec{L}$ is perpendicular to the projection of the missile axis and since it lies in the Y'Z' plane it is also perpendicular to X'. $\vec{L}$ is perpendicular to the plane determined by the missile axis and the line of sight and is therefore perpendicular to the missile axis itself. The direction cosines of $\vec{L}$ in the X'Y'Z' system are 0, $\cos V_i$, $-\sin V_i$ and the direction cosines (u_i, v_i, w_i) of $\vec{L}$ in the XYZ system are obtained as follows.

$$\begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} = \begin{pmatrix} \cos \alpha_i \cos \varepsilon_i & -\sin \alpha_i & -\cos \alpha_i \sin \varepsilon_i \\ \sin \alpha_i \cos \varepsilon_i & \cos \alpha_i & -\sin \alpha_i \sin \varepsilon_i \\ \sin \varepsilon_i & 0 & \cos \varepsilon_i \end{pmatrix} \begin{pmatrix} 0 \\ \cos V_i \\ -\sin V_i \end{pmatrix}$$

If a_m, b_m, c_m denote the direction cosines of the missile axis then

$$a_m u_i + b_m v_i + c_m w_i = \cos \phi_i \tag{1}$$

ϕ is the angle between the missile axis and $\vec{L}$. Therefore, equation (1) becomes

$$a_m u_i + b_m v_i + c_m w_i = 0 \tag{2}$$

The function to be minimized is the sum of the squares of the residuals. The residuals are the $\cos \phi_i$, which would be zero if no errors were present.

The least squares sum to be minimized becomes

$$S = \Sigma(0 - a_m u_i - b_m v_i - c_m w_i)^2 \tag{3}$$

and

$$\frac{\partial S}{\partial a_m} = 0 = 2(a_m \Sigma u_i^2 + b_m \Sigma u_i v_i + c_m \Sigma u_i w_i)$$

$$\frac{\partial S}{\partial b_m} = 0 = 2(a_m \Sigma u_i v_i + b_m \Sigma v_i^2 + c_m \Sigma v_i w_i)$$

$$\frac{\partial S}{\partial c_m} = 0 = 2(a_m \Sigma u_i w_i + b_m \Sigma v_i w_i + c_m \Sigma w_i^2)$$

The simultaneous equations are represented by

$$a_m \Sigma u_i^2 + b_m \Sigma u_i v_i + c_m \Sigma u_i w_i = 0$$

$$a_m \Sigma u_i v_i + b_m \Sigma v_i^2 + c_m \Sigma v_i w_i = 0$$

$$a_m \Sigma u_i w_i + b_m \Sigma v_i w_i + c_m \Sigma w_i^2 = 0$$

Any combination of two of the above equations is sufficient to yield a non-trivial solution for a_m, b_m, c_m provided that at least two $\vec{L}$ vectors are not parallel.

The proportional relationship between the direction cosines in any one of the three possible solutions is

$$a_m:b_m:c_m = p:q:r$$

Solving the first two equations, p, q and r become

$$p = \begin{vmatrix} \Sigma u_i v_i & \Sigma u_i w_i \\ \Sigma v_i^2 & \Sigma v_i w_i \end{vmatrix}$$

$$q = - \begin{vmatrix} \Sigma u_i^2 & \Sigma u_i w_i \\ \Sigma u_i v_i & \Sigma v_i w_i \end{vmatrix}$$

$$r = \begin{vmatrix} \Sigma u_i^2 & \Sigma u_i v_i \\ \Sigma u_i v_i & \Sigma v_i^2 \end{vmatrix}$$

Then if $D = \pm \sqrt{p^2 + q^2 + r^2}$, p, q and r are direction numbers,

$$a_m = \frac{p}{D}$$

$$b_m = \frac{q}{D}$$

$$c_m = \frac{r}{D}$$

Since D has both a positive and negative value it is necessary to determine the correct signs of a_m, b_m, c_m. Assuming that D is positive the correct signs may be readily found.

The direction cosines of one of the missile axis projections give rise to the following equation:

$$l_i\, a_m + m_i\, b_m + n_i\, c_m = \cos \beta_i \tag{4}$$

where β_i is the angle between the missile axis and its projection in the film plane. β_i is always an acute angle. Therefore

$$l_i\, a_m + m_i\, b_m + n_i\, c_m > 0 \tag{5}$$

In the event that $\cos \beta_i$ is close to zero this equation is invalid for that particular station. Therefore, it is advisable to check at least two of the stations in the following manner.

If $(l_i a_m + m_i b_m + n_i c_m) > 0$ for both stations then the signs of the direction cosines are correct.

If $(l_i a_m + m_i b_m + n_i c_m) < 0$ for both stations then the signs of a_m, b_m, c_m must be changed.

If the $(l_i a_m + m_i b_m + n_i c_m)$ from the two stations are opposite in signs, the station with the larger absolute value of $(l_i a_m + m_i b_m + n_i c_m)$ is used to determine the correct signs of the direction cosines, by use of the above logic.

The approximate attitude angles may now be obtained by using the direction cosines of the missile axis.

$$\alpha_A^* = \tan^{-1} \frac{b_m}{a_m} \tag{6}$$

$$\epsilon_A^* = \tan^{-1} \frac{c_m}{[a^2_m + b^2_m]^{\frac{1}{2}}} \tag{7}$$

We now attempt to compute a "most probable attitude". The function to be minimized is the sum of the squares of the angular residuals. The angular residuals are the differences between the measured V-angles and the most probable V-angles. The most probable V-angle may be expressed as a function of the most probable missile attitude and station location.

Since a_m, b_m, c_m are the direction cosines of the missile axis in the XYZ-system, the direction cosines of the missile axis in the X'Y'Z'-system may be obtained by the following rotation:

$$\begin{pmatrix} a'_m \\ b'_m \\ c'_m \end{pmatrix} = \begin{pmatrix} \cos \alpha_i \cos \epsilon_i & \sin \alpha_i \cos \epsilon_i & \sin \epsilon_i \\ -\sin \alpha_i & \cos \alpha_i & 0 \\ -\cos \alpha_i \sin \epsilon_i & -\sin \alpha_i \sin \epsilon_i & \cos \epsilon_i \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix} \tag{8}$$

and it is easily seen that

$$\tan V_i^* = \frac{b'_m}{c'_m} = \frac{-a_m \sin \alpha_i + b_m \cos \alpha_i}{-a_m \cos \alpha_i \sin \epsilon_i - b_m \sin \alpha_i \sin \epsilon_i + c_m \cos \epsilon_i} \tag{9}$$

where V_i^* is the most probable V-angle.

The most probable missile attitude is an approximate attitude plus a small change in attitude. By the use of a Taylor's series expansion, where A equals the approximate attitude and ΔA equals the change in attitude, the expression for the most probable V-angle becomes

$$f(A + \Delta A) = f(A) + \Delta A\ f'(A) + \frac{\Delta A^2}{2} f''(A) + \cdots \tag{10}$$

Since ΔA is a very small change all higher order terms may be discarded as negligible and the function to be minimized becomes

$$S = \Sigma[V_i - f_i\ (A) - \Delta A f_i'\ (A)]^2 \tag{11}$$

If we assume $\alpha_A{}^\circ$, $\varepsilon_A{}^\circ$ to be close approximations of the attitude, then from equation (9)

$$f_i(A) = V_i^o$$

$$f_i(A) = \tan^{-1} \frac{-a_m \sin \alpha_i + b_m \cos \alpha_i}{-a_m \cos \alpha_i \sin \varepsilon_i - b_m \sin \alpha_i \sin \varepsilon_i + c_m \cos \varepsilon_i} \tag{12}$$

and

$$\Delta A f_i'\ (A) = \Delta\alpha \left(\frac{\partial V_i{}^\circ}{\partial \alpha_A{}^\circ}\right) + \Delta\varepsilon \left(\frac{\partial V_i{}^\circ}{\partial \varepsilon_A{}^\circ}\right) \tag{13}$$

where

$$\frac{\partial V_i{}^\circ}{\partial \alpha_A{}^\circ} = \sin V_i{}^\circ\ [\cos V_i{}^\circ \cot (\alpha_A{}^\circ - \alpha_i) - \sin V_i{}^\circ \sin \varepsilon_i] \tag{14}$$

$$\frac{\partial V_i{}^\circ}{\partial \varepsilon_A{}^\circ} = \frac{-\sin^2 V_i{}^\circ \cos \varepsilon_i}{\sin (\alpha_A{}^\circ - \alpha_i) \cos^2 \varepsilon_A{}^\circ} \tag{15}$$

α_i, ε_i are azimuth and elevation angles of the missile with respect to the i^{th} station.

V_i is the measured V-angle of the i^{th} station. Then substituting equations (12) and (13) in equation (11)

$$S = \Sigma \left[\left(V_i - V_i^{\circ} \right) - \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \Delta\alpha - \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) \Delta\varepsilon \right]^2 \tag{16}$$

and

$$\frac{\partial S}{\partial \Delta\alpha} = 0 = 2\,\Sigma \left[\left(V_i - V_i^{\circ} \right) \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) - \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right)^2 \Delta\alpha - \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) \Delta\varepsilon \right]$$

$$\frac{\partial S}{\partial \Delta\varepsilon} = 0 = 2\,\Sigma \left[\left(V_i - V_i^{\circ} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) - \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) \Delta\alpha - \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right)^2 \Delta\varepsilon \right]$$

The simultaneous equations are

$$\Delta\alpha\Sigma \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right)^2 + \Delta\varepsilon\Sigma \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) = \Sigma \left(V_i - V_i^{\circ} \right) \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \tag{17}$$

$$\Delta\alpha\Sigma \left(\frac{\partial V_i^{\circ}}{\partial \alpha_A^{\circ}} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) + \Delta\varepsilon\Sigma \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right)^2 = \Sigma \left(V_i - V_i^{\circ} \right) \left(\frac{\partial V_i^{\circ}}{\partial \varepsilon_A^{\circ}} \right) \tag{18}$$

Solving these equations for $\Delta\alpha$ and $\Delta\varepsilon$, the most probable attitude becomes

$$\alpha_A = \alpha_A^{\circ} + \Delta\alpha \tag{19}$$

$$\varepsilon_A = \varepsilon_A^{\circ} + \Delta\varepsilon \tag{20}$$

α_A and ε_A then become the new approximation of the attitude and a new most probable attitude is computed.

This process is repeated until

$\Delta\alpha \leq .01$

$\Delta c \leq .01$

Since $\Delta\alpha$ and $\Delta\varepsilon$ are small the sum of the squares of the residuals now becomes

$$S = \Sigma(V_i - V_i^{\circ})^2 \tag{21}$$

We now proceed to the problem of locating and rejecting V-angles having errors greater than the errors to be expected.

Using the F test

$$\left(\frac{\hat{\sigma}_1}{\hat{\sigma}_2}\right)^2 = F \tag{22}$$

where

$$\hat{\sigma}_1^2 = \frac{(V_i - V_i^*)^2}{1}$$

$$\hat{\sigma}_2^2 = \frac{S-(V_i - V_i^*)^2}{N-1}$$

N = Number of stations used in the solution.

Substituting in equation (22)

$$\frac{\dfrac{(V_i - V_i^*)^2}{1}}{\dfrac{S-(V_i - V_i^*)^2}{N-1}} = F$$

If one or more $(V_i - V_i^*)^2 > F\left[\dfrac{S-(V_i - V_i^*)^2}{N-1}\right]$

the station with the largest $(V_i - V_i^*)^2$ is rejected and a new attitude is computed. This process is repeated until all

$$(V_i - V_i^*)^2 < F\left[\frac{S-(V_i - V_i^*)^2}{N-1}\right]$$

An approximate value of F computed for the degrees of freedom encountered in this solution is

$$F \cong 17.44 + 15.792\left(\frac{4-N}{N-1}\right)$$

At this point a further check is made. If the variance of the V-angle is greater than the maximum allowable value, the station with the largest $(V_i - V_i^*)^2$ is rejected and a new attitude is computed.

The variances of the V-angles are found by

$$\sigma_V^2 = \frac{S}{N-2} \tag{23}$$

and the variances of the attitude are then computed using the σ_V^2 above and the co-factors of the least squares determinant formed from equations (17) and (18).

$$\sigma\alpha_A^2 = \frac{A_{11}}{\Delta}\,\sigma_V^2 \tag{24}$$

$$\sigma\epsilon_A^2 = \frac{A_{22}}{\Delta}\,\sigma_V^2 \tag{25}$$

where Δ is the value of the determinant.

Definitions of symbols used in the solution

α_i, ϵ_i - Azimuth and Elevation angles of the missile with respect to the i^{th} station.

X_i, Y_i, Z_i - Coordinates of the i^{th} station (same system as the missile coordinates) with respect to the launcher.

X_m, Y_m, Z_m - Coordinates of the missile.

V_i - Observed V-angle from the i^{th} station.

l_i, m_i, n_i - Direction cosines of the image vector in the XYZ-system.

$\vec{L}$ - Vector perpendicular to the missile axis.

u_i, v_i, w_i - Direction cosines of $\vec{L}$ in the XYZ-system.

a_m, b_m, c_m - Direction cosines of the missile axis in the XYZ-system.

a_m', b_m', c_m' - Direction cosines of the missile axis in the X'Y'Z' system.

β_i - Angle between the missile axis and its projection in the film plane.

$\alpha_A{}^*$, $\epsilon_A{}^*$ - Approximate attitude angles.

$\Delta\alpha$, $\Delta\epsilon$ - Change in the attitude angles.

$V_i{}^*$ - A function of the approximate attitude.

α_A, ε_A - Final attitude.

σ_V^2 - variance of the V-angles.

$\sigma\alpha_A^2$, $\sigma\varepsilon_A^2$ - variances of the attitude.

COMPUTATIONAL ORDER OF FORMULAE

$$\alpha_i = \tan^{-1}\left(\frac{Y_m - Y_i}{X_m - X_i}\right) \tag{1}$$

$$\varepsilon_i = \tan^{-1}\frac{(Z_m - Z_i)}{[(X_m - X_i)^2 + (Y_m - Y_i)^2]^{\frac{1}{2}}} \tag{2}$$

Solve for l_i, m_i, and n_i from

$$\begin{pmatrix} l_i \\ m_i \\ n_i \end{pmatrix} = \begin{pmatrix} \cos\alpha_i\cos\varepsilon_i & -\sin\alpha_i & -\cos\alpha_i\sin\varepsilon_i \\ \sin\alpha_i\cos\varepsilon_i & \cos\alpha_i & -\sin\alpha_i\sin\varepsilon_i \\ \sin\varepsilon_i & 0 & \cos\varepsilon_i \end{pmatrix} \begin{pmatrix} 0 \\ \sin V_i \\ \cos V_i \end{pmatrix} \tag{3}$$

and u_i, v_i, and w_i

$$\begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} = \begin{pmatrix} \cos\alpha_i\cos\varepsilon_i & -\sin\alpha_i & -\cos\alpha_i\sin\varepsilon_i \\ \sin\alpha_i\cos\varepsilon_i & \cos\alpha_i & -\sin\alpha_i\sin\varepsilon_i \\ \sin\varepsilon_i & 0 & \cos\varepsilon_i \end{pmatrix} \begin{pmatrix} 0 \\ \cos V_i \\ -\sin V_i \end{pmatrix} \tag{4}$$

$$p = \begin{vmatrix} \Sigma u_i v_i & \Sigma u_i w_i \\ \Sigma v_i^2 & \Sigma v_i w_i \end{vmatrix} \tag{5}$$

$$q = -\begin{vmatrix} \Sigma u_i^2 & \Sigma u_i w_i \\ \Sigma u_i v_i & \Sigma v_i w_i \end{vmatrix} \tag{6}$$

$$r = \begin{vmatrix} \Sigma u_i^2 & \Sigma u_i v_i \\ \Sigma u_i v_i & \Sigma v_i^2 \end{vmatrix} \tag{7}$$

$$D = \sqrt{p^2 + q^2 + r^2} \tag{8}$$

$$a_m = \frac{p}{D} \tag{9}$$

$$b_m = \frac{q}{D} \tag{10}$$

$$c_m = \frac{r}{D} \tag{11}$$

Test

$$\left.\begin{array}{ll} l_1\, a_m + m_1\, b_m + n_1\, c_m > 0 & (1) \\ l_2\, a_m + m_2\, b_m + n_2\, c_m > 0 & (2) \end{array}\right\} \tag{12}$$

for any two stations.

If both (1) and (2) are >0, a_m, b_m, c_m have correct signs.

If both (1) and (2) are <0, change signs of a_m, b_m, c_m.

If (1) and (2) are opposite in sign take the one with larger absolute value of $l_i\, a_m + m_i\, b_m + n_i\, c_m$ and use the previous logic to check signs.

Solve

$$\begin{pmatrix} a'_m \\ b'_m \\ c'_m \end{pmatrix} = \begin{pmatrix} \cos\alpha_i \cos\epsilon_i & \sin\alpha_i \cos\epsilon_i & \sin\epsilon_i \\ -\sin\alpha_i & \cos\alpha_i & 0 \\ -\cos\alpha_i \sin\epsilon_i & -\sin\alpha_i \sin\epsilon_i & \cos\epsilon_i \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix}$$

then $V_i^\circ = \tan^{-1}\left(\frac{b'_m}{c'_m}\right)$ (13)

$$\frac{\partial V_i^\circ}{\partial \alpha_A^\circ} = \sin V_i^\circ \left[\cos V_i^\circ \cot(\alpha_A^\circ - \alpha_i) - \sin V_i^\circ \sin \epsilon_i\right] \tag{14}$$

$$\frac{\partial V_i^\circ}{\partial \epsilon_A^\circ} = \frac{-\sin^2 V_i^\circ \cos\epsilon_i}{\sin(\alpha_A^\circ - \alpha_i)\cos^2\epsilon_A^\circ} \tag{15}$$

Solve for $\Delta\alpha$ and $\Delta\varepsilon$ in the following equations

$$\Delta\alpha\Sigma\left(\frac{\partial V_i^\circ}{\partial\alpha_A^\circ}\right)^2 + \Delta\varepsilon\Sigma\left(\frac{\partial V_i^\circ}{\partial\varepsilon_A^\circ}\right)\left(\frac{\partial V_i^\circ}{\partial\alpha_A^\circ}\right) = \Sigma(V_i - V_i^\circ)\left(\frac{\partial V_i^\circ}{\partial\alpha_A^\circ}\right) \tag{16}$$

$$\Delta\alpha\Sigma\left(\frac{\partial V_i^\circ}{\partial\alpha_A^\circ}\right)\left(\frac{\partial V_i^\circ}{\partial\varepsilon_A^\circ}\right) + \Delta\varepsilon\Sigma\left(\frac{\partial V_i^\circ}{\partial\varepsilon_A^\circ}\right)^2 = \Sigma(V_i - V_i^\circ)\left(\frac{\partial V_i^\circ}{\partial\varepsilon_A^\circ}\right) \tag{17}$$

Compute

$$\alpha_A = \alpha_A^\circ + \Delta\alpha \tag{18}$$

$$\varepsilon_A = \varepsilon_A^\circ + \Delta\varepsilon \tag{19}$$

α_A, ε_A now become the new approximations. Equations (13) thru (19) are repeated until

$$\Delta\alpha \leq .01$$

$$\Delta\varepsilon \leq .01$$

$$S = \Sigma(V_i - V_i^\circ)^2 \tag{20}$$

Test if

$$(V_i - V_i^\circ)^2 > F\left[\frac{S-(V_i - V_i^\circ)^2}{N-1}\right] \tag{21}$$

If one or more $(V_i - V_i^\circ)^2 > F\left[\frac{S-(V_i - V_i^\circ)^2}{N-1}\right]$

reject the station with the largest $(V_i - V_i^\circ)^2$ and recompute the attitude until all

$$(V_i - V_i^\circ)^2 < F\,\frac{S-(V_i - V_i^\circ)^2}{N-1}$$

$$F \cong 17.44 + 15.792\left(\frac{4-N}{N-1}\right) \tag{22}$$

If $\left(\frac{S}{N-2}\right) > 225$, reject the station with the largest $(V_i - V_i^\circ)^2$ and recompute the attitude.

Compute

$$\sigma_V^2 = \frac{S}{N-2} \tag{23}$$

$$\sigma \alpha_A^2 = \frac{A_{11}}{\Delta} \sigma^2_V \tag{24}$$

$$\sigma^2 c_A = \frac{A_{22}}{\Delta} \sigma^2_V \tag{25}$$

where A_{11} and A_{22} are the cofactors of the least squares determinant formed from equations (16) and (17) and Δ is the value of the determinant.

APPENDIX I. F-test for N-Station Attitude

The value of F for the F-test for the N-station attitude program is an approximate value found by linear interpolation for values of F in the interval of 3 to 8 degrees of freedom at the 97.5% confidence level.

$F_3 = 17.44$

$F_8 = 7.57$

$$F_{DF} = 17.44 + (7.57 - 17.44)\frac{\left(\frac{1}{DF} - \frac{1}{3}\right)}{\left(\frac{1}{8} - \frac{1}{3}\right)}$$

$$= 17.44 + \frac{(-9.87)}{(-5/24)}\left(\frac{3 - DF}{3\ DF}\right)$$

$$= 17.44 + \frac{(9.87)(24)}{(5)(3)}\left(\frac{3 - DF}{DF}\right)$$

$$= 17.44 + 15.792\left(\frac{3 - DF}{DF}\right)$$

DF in this case is N-1.

Therefore,

$$F_{DF} = 17.44 + 15.792\left(\frac{4-N}{N-1}\right) \text{ where N = no. of stations used.}$$

The difference between the computed F using this equation and F found from the tables is shown below:

DF	N	F computed	F table	
2	3	25.08	38.51	
3	4	17.44	17.44	Most of the data will fall in here.
4	5	13.49	12.22	
5	6	11.13	10.01	
6	7	9.55	8.81	
7	8			
8	9	7.57	7.57	
9	10			
10	11	6.39	6.94	
17	18	4.44	6.04	

REFERENCES

1. Attitude and Yaw Reductions of Projectiles in Free Flight, G.R. Trimble, Jr., Ballistic Research Laboratories Report No. 774, Aberdeen Proving Ground, Md.

2. The Calculus of Observations, Wittaker and Robinson, D. Van Nostrand Co., Inc, New York.

C. ATTITUDE DATA

III Little Joe Primary Paint Pattern Attitude

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LITTLE JOE

Primary Paint Pattern Attitude

There are two opposing pairs of paint markers on the base of the Little Joe command module. That diameter of the base which passes through the centers of either pair of paint markers is designated as the Primary Paint Pattern. In Figure 1, either AB or CD may be chosen as the primary paint pattern.

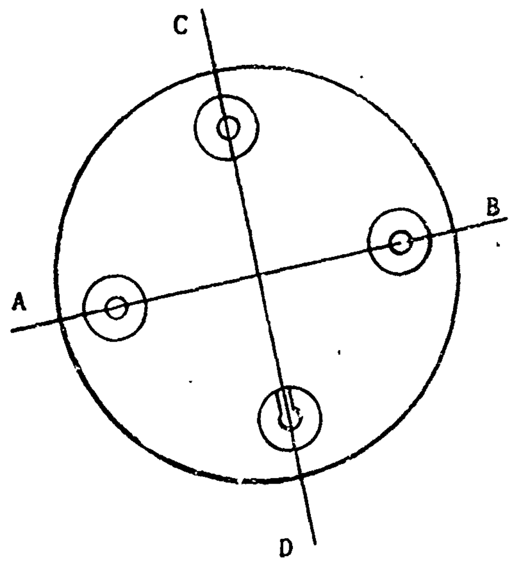

FIGURE 1

The attitude of the primary paint pattern on the base may be used by the project (NASA) to obtain roll data. If, for some reason, the attitude of this paint pattern is not available it is possible to obtain the data by knowing the following three things: (1) The attitude of any other paint pattern on the module surface, (2) The attitude of the module, and (3) The true relationship of the paint pattern being used to the primary paint pattern.

Assuming we know the attitude (α_s, ε_s), in the x, y, z system, of a paint stripe on the side of the module, the attitude of the module axis (α_m, ε_m) in the x, y, z system, and the actual angle ($\Delta\alpha$) between the strip projected onto the base and the primary paint pattern, we may proceed as follows:

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If the module axis is rotated counter-clockwise about the z axis through the azimuth angle (α_m) of the module and then counter-clockwise about the y axis through an angle of ($90° - \varepsilon_m$) then the module is in a vertical position with the module axis parallel to the z axis and the base parallel to the xy plane. As the axis is being rotated through these angles, the paint stripe is also being rotated through the same angles. Therefore, we can obtain the direction cosines of the stripe when the module is in the vertical position by rotating the direction cosines of the stripe

$$a_s = \cos \alpha_s \cos \varepsilon_s$$

$$b_s = \sin \alpha_s \cos \varepsilon_s$$

$$c_s = \sin \varepsilon_s$$

through the following rotational matrix

$$\begin{pmatrix} a'_s \\ b'_s \\ c'_s \end{pmatrix} = \begin{pmatrix} \cos \alpha_m \sin \varepsilon_m & \sin \alpha_m \sin \varepsilon_m & -\cos \varepsilon_m \\ -\sin \alpha_m & \cos \alpha_m & 0 \\ \cos \alpha_m \cos \varepsilon_m & \sin \alpha_m \cos \varepsilon_m & \sin \varepsilon_m \end{pmatrix} \begin{pmatrix} a_s \\ b_s \\ c_s \end{pmatrix}$$

With the module in this vertical position we project the paint stripe onto the xy plane. The direction cosines of this projection are

$$a_p = \frac{a'_s}{[(a'_s)^2 + (b'_s)^2]^{\frac{1}{2}}}$$

$$b_p = \frac{b'_s}{[(a'_s)^2 + (b'_s)^2]^{\frac{1}{2}}}$$

$$c_p = 0$$

If the projection of the stripe onto the base is not coincident with the primary paint pattern then the direction cosines of the projection must be rotated through the angle ($\Delta\alpha$) as follows:

$$\begin{pmatrix} a'_p \\ b'_p \\ c'_p \end{pmatrix} = \begin{pmatrix} \cos \Delta\alpha & \sin \Delta\alpha & 0 \\ -\sin \Delta\alpha & \cos \Delta\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_p \\ b_p \\ c_p \end{pmatrix}$$

The direction cosines of the projection of the stripe coincident with the primary paint pattern are then rotated back to the original position by the following matrix:

$$\begin{pmatrix} a_{pb} \\ b_{pb} \\ c_{pb} \end{pmatrix} = \begin{pmatrix} \cos\alpha_m \sin\varepsilon_m & -\sin\alpha_m & \cos\alpha_m \cos\varepsilon_m \\ \sin\alpha_m \sin\varepsilon_m & \cos\alpha_m & \sin\alpha_m \cos\varepsilon_m \\ -\cos\alpha_m & 0 & \sin\varepsilon_m \end{pmatrix} \begin{pmatrix} a_p' \\ b_p' \\ c_p' \end{pmatrix}$$

The attitude of stripe projected onto the base of the module and coinciding with the primary paint pattern is then given by:

$$\alpha_A = \tan^{-1} \frac{b_{pb}}{a_{pb}}$$

$$\varepsilon_A = \tan^{-1} \frac{c_{pb}}{(a_{pb}^2 + b_{pb}^2)^{1/2}}$$

C. ATTITUDE DATA

IV Little Joe Single Station Primary Paint Pattern Solution

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LITTLE JOE

Single Station Primary Paint Pattern Solution

The following is a method of determining the attitude data of the primary paint pattern on the base of the command module when the V-angle of the paint pattern is available from only one camera.

The position data of the module, the attitude data of the module and the V-angles of the primary paint pattern are required for this reduction.

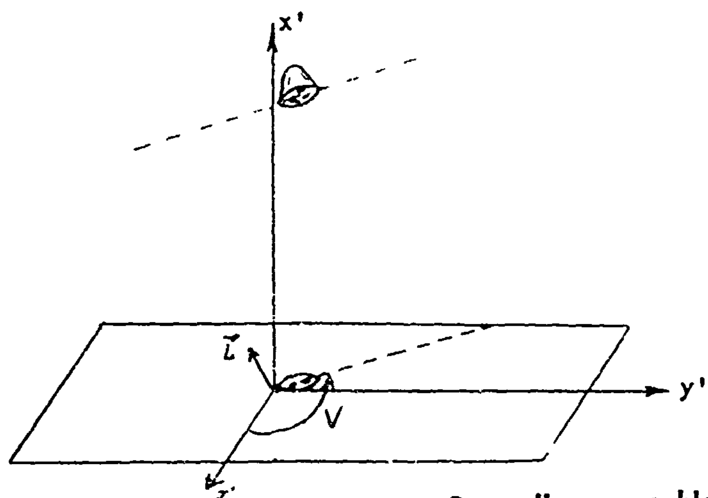
The coordinate system used in the reduction of the position data is defined as the x, y, z system. In deriving the equations for this attitude reduction it is necessary to set up a new coordinate system x', y', z'. The x' axis is positive along the line of sight from the camera to the module. The y' axis lies in the film plane perpendicular to the x' axis and is directed positive to the right and parallel to the xy plane. The z' axis also lies in the film plane, perpendicular to the x'y' plane, and positive up.

In this coordinate system the direction cosines of the vector lying along the paint pattern image are sine and cosine functions of the V-angles. The V-angle is measured clockwise from the z' axis to the paint pattern image. The direction cosines may be rotated into the x, y, z system by the following rotational matrix.

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\varepsilon & -\sin\alpha & -\cos\alpha\sin\varepsilon \\ \sin\alpha\cos\varepsilon & \cos\alpha & -\sin\alpha\sin\varepsilon \\ \sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix} \begin{pmatrix} 0 \\ \sin V \\ \cos V \end{pmatrix}$$

l, m, n are the direction cosines of the image vector in the x, y, z system. α, ε are the azimuth and elevation angles of the module with respect to the camera.

The following figure represents the situation that exists at the station:



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$\vec{L}$ is perpendicular to the image of the paint pattern. Since it lies in the y'z' plane it is also perpendicular to x'. $\vec{L}$ is perpendicular to the plane determined by the paint pattern and the line of sight and therefore is perpendicular to the paint pattern itself. The direction cosines of $\vec{L}$ in the x', y', z' system are 0, cos V, -sin V. The direction cosines (u, v, w) of $\vec{L}$ in the x, y, z system are obtained as follows:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\varepsilon & -\sin\alpha & -\cos\alpha\sin\varepsilon \\ \sin\alpha\cos\varepsilon & \cos\alpha & -\sin\alpha\sin\varepsilon \\ \sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix} \begin{pmatrix} 0 \\ \cos V \\ -\sin V \end{pmatrix}$$

If a_p, b_p, c_p denote the direction cosines of the paint pattern, then

$$a_p\, u + b_p\, v + c_p\, w = \cos\phi \tag{1}$$

ϕ is the angle between the paint pattern and $\vec{L}$. Therefore, equation (1) becomes

$$a_p\, u + b_p\, v + c_p\, w = 0 \tag{2}$$

a_m, b_m, c_m are the direction cosines of the module axis and are computed from the attitude of the module. Since the module axis is perpendicular to the paint pattern then

$$a_p\, a_m + b_p\, b_m + c_p\, c_m = 0 \tag{3}$$

. also know that

$$a_p\, a_p + b_p\, b_p + c_p\, c_p = 1 \tag{4}$$

Therefore we have three simultaneous equations with three unknowns.

$$a_p\, a_p + b_p\, b_p + c_p\, c_p = 1$$

$$a_p\, u + b_p\, v + c_p\, u = 0$$

$$a_p\, a_m + b_p\, b_m + c_p\, c_m = 0$$

Solving these equations we obtain

$$a_p = \frac{(v\ c_m - b_m\ w)}{D} = \frac{p}{D}$$

$$b_p = \frac{-(u\ c_m - a_m\ w)}{D} = \frac{q}{D}$$

$$c_p = \frac{(u\ b_m - a_m\ v)}{D} = \frac{r}{D}$$

where $D = \pm \sqrt{p^2 + q^2 + r^2}$

Since D has both a positive and negative value it is necessary to determine the correct signs of a_p, b_p, c_p. Assuming that D is positive the correct signs may be readily found as follows:

The direction cosines of the paint pattern image give rise to the equation:

$$l\ a_p + m\ b_p + n\ c_p = \cos\ \beta$$

where β is the angle between the paint pattern and its image in the film plane. β is always an acute angle. Therefore

$$l\ a_p + m\ b_p + n\ c_p > 0$$

If $l\ a_p + m\ b_p + n\ c_p < 0$, then the signs of a_p, b_p, c_p must be changed.

The attitude angles are then obtained using the direction cosines of the paint pattern.

$$\alpha_A = \tan^{-1} \frac{b_p}{a_p}$$

$$\epsilon_A = \tan^{-1} \frac{c_p}{(a_p^2 + b_p^2)^{\frac{1}{2}}}$$

C. ATTITUDE DATA

V Angle of Attack

ANGLE OF ATTACK

Introduction

The angle of attack is defined as the angle between the longitudinal axis of the missile and a line along the direction of motion.

The cosine of the angle of attack is the dot product of a unit vector lying along the missile axis and a unit vector lying along the direction of motion. A unit vector lying along a line has as its components the direction cosines of that line. By knowing the angle between a line and the XY-plane (elevation angle) and the angle between the projection of that line in the XY-plane and the X-axis (azimuth angle) a set of direction cosines may be computed.

A unit vector lying along the missile axis has as its direction cosines the direction cosines of the missile attitude. The direction cosines for a line along the direction of motion are the direction cosines of the velocity vector.

Mathematical Procedure:

Any available missile attitude data and trajectory data may be used to compute the angle of attack data.

The angle of attack will always be between 0 and 180 degrees. By considering the sign of the sine and cosine, the proper quadrant is determined.

Definitions:

α_A = azimuth attitude angle

ε_A = elevation attitude angle

θ = azimuth trajectory angle

ϕ = elevation trajectory angle

A = angle of attack

α, ε, θ and ϕ must all be in the same coordinate system.

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The direction cosines of the missile are:

$$a_m = \cos \alpha_A \cos \varepsilon_A$$

$$b_m = \sin \alpha_A \cos \varepsilon_A$$

$$c_m = \sin \varepsilon_A$$

The direction cosines along the direction of motion are:

$$a_D = \cos \theta \cos \phi$$

$$b_D = \sin \theta \cos \phi$$

$$c_D = \sin \phi$$

The dot product of the two vectors is computed as follows:

$$\cos A = a_m a_D + b_m b_D + c_m c_D$$

and

$$A = \cos^{-1} (a_m a_D + b_m b_D + c_m c_D)$$

The sine A is computed as follows:

$$\mathrm{Sin} \frac{A}{2} = \left(\frac{1 - \cos A}{2}\right)^{\frac{1}{2}} = \left(\frac{1 - a_m a_D - b_m b_D - c_m c_D}{2}\right)^{\frac{1}{2}}$$

$$\mathrm{Cos} \frac{A}{2} = \left(\frac{1 + \cos A}{2}\right)^{\frac{1}{2}} = \left(\frac{1 + a_m a_D + b_m b_D + c_m c_D}{2}\right)^{\frac{1}{2}}$$

and

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\tan A = \frac{\sin A}{\cos A}$$

C. ATTITUDE DATA

VI Aspect Angle

ASPECT ANGLE

Introduction

The aspect angle is defined as the angle between the longitudinal axis of the missile and the line of sight from the missile to the object whose aspect angle is being computed. (See Fig. 1)

The cosine of the aspect angle is the dot product of a unit vector lying along the missile axis and a unit vector lying along the line of sight. The vector along the missile axis is directed from tail to nose and the vector along the line of sight is directed toward the object whose aspect angle is being measured.

A unit vector lying along a line has as its components the direction cosines of that line. By knowing the angle between a line and the XY-plane (elevation angle) and the angle between the projection of that line in the XY-plane and the X-axis (azimuth angle) a set of direction cosines may be computed.

A set of direction cosines for the unit vector lying along the line of sight may be computed if the position of the missile and the position of the object whose aspect angle is being measured are known, or if the "look angles" are known.

A unit vector lying along the missile axis has as its direction cosines the direction cosines of the missile attitude.

Mathematical Procedure

Any available missile attitude data and either position data or "look angle" data may be used to compute the aspect angle data.

The aspect angle will always be between 0 and 180 degrees. By considering the sign of the sine and cosine, the proper quadrant is determined.

Definition

X_m, Y_m, Z_m = Coordinates of missile position

X_c, Y_c, Z_c = Coordinates of the camera or object whose aspect angle is being computed

α_A = azimuth attitude angle

ε_A = elevation attitude angle

A_L = Aspect angle

These data are in the same coordinate system.

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Compute:

The direction cosines of the missile axis are:

$$a_m = \cos \alpha_A \sin \varepsilon_A$$

$$b_m = \sin \alpha_A \cos \varepsilon_A$$

$$c_m = \sin \varepsilon_A$$

The direction cosines along the line of sight are:

$$a_s = \frac{X_c - X_m}{R}$$

$$b_s = \frac{Y_c - Y_m}{R}$$

$$c_s = \frac{Z_c - Z_m}{R}$$

where

$$R = [(X_c - X_m)^2 + (Y_c - Y_m)^2 + (Z_c - Z_m)^2]^{1/2}$$

The dot product of the two vectors is computed as follows:

$$\cos A_L = a_m a_s + b_m b_s + c_m c_s \qquad (1)$$

$$A_L = \cos^{-1} (a_m a_s + b_m b_s + c_m c_s)$$

The sine A_L is computed as follows:

$$\sin \frac{A_L}{2} = \left(\frac{1 - \cos_{AL}}{2}\right)^{1/2} = \left(\frac{1 - a_m a_s - b_m b_s - c_m c_s}{2}\right)^{1/2}$$

$$\cos \frac{A_L}{2} = \left(\frac{1 + \cos_{AL}}{2}\right)^{1/2} = \left(\frac{1 + a_m a_s + b_m b_s + c_m c_s}{2}\right)^{1/2}$$

and

$$\sin A_L = 2 \sin \frac{A_L}{2} \cos \frac{A_L}{2}$$

$$\tan A_L = \frac{\sin A_L}{\cos A_L}$$

If "look angles" are known the direction cosines of the line of sight become:

$$a_s = -\cos \alpha_L \sin \epsilon_L$$

$$b_s = -\sin \alpha_L \cos \epsilon_L$$

$$c_s = -\sin \epsilon_L$$

where

$$\alpha_L, \epsilon_L = \text{"look angles"}$$

These direction cosines are then substituted in equation (1) to determine the aspect angle.

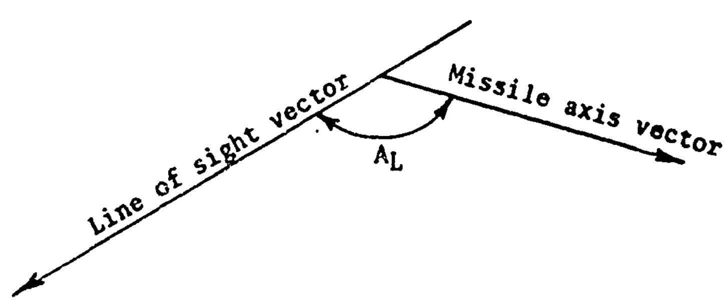

FIGURE 1

C. ATTITUDE DATA

VII Ground Distance and Total Distance

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GROUND DISTANCE AND TOTAL DISTANCE

Introduction

Ground distance is defined as the cumulative distance a missile travels from a given point up to a time T_i, as projected on the XY-plane.

Total distance is the cumulative distance a missile travels from a given point up to a time T_i.

Mathematical Procedure:

Any available position data are used to obtain ground distance or total distance.

Definitions:

X_j, Y_j, Z_j = coordinates of the missile at point j

D_{G_i} = Ground distance traveled

D_{T_i} = Total distance traveled

Ground distance:

$$D_{G_i} = \sum_{j=1}^{i} [(X_j - X_{j-1})^2 + (Y_j - Y_{j-1})^2]^{\frac{1}{2}}$$

Total distance:

$$D_{T_i} = \sum_{j=1}^{i} [(X_j - X_{j-1})^2 + (Y_j - Y_{j-1})^2 + (Z_j - Z_{j-1})^2]^{\frac{1}{2}}$$

If there is a break in position data, the distance traveled up to the break is totaled and another summation is started again at the next available position.

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D. VELOCITY AND ACCELERATION

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D. VELOCITY AND ACCELERATION

1 Smoothed Positions, Velocity and Acceleration (Moving Arc)

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SMOOTHED POSITIONS, VELOCITY & ACCELERATION (MOVING ARC)

Introduction

Because of the fact that all physical measurements contain some random errors (or noise), it is usually desirable to compute from observed data an estimate of the data which would have been observed by a noise-free measuring system. This process of minimizing the errors in observations, called "smoothing",* can be done in many ways. The method discussed in this report, "Least Squares Moving Arc Smoothing", is used in DRD to compute smoothed position data from observations. These smoothed positions are then differentiated to obtain velocities, and the velocities differentiated to obtain accelerations. Error estimates of the smoothed data and derivatives are computed in the form of standard deviations for each point.

*This process is also known as "filtering" or "adjustment of data".

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Moving Arc Smoothing

The Least Squares Moving Arc Smoothing technique assumes that for a short interval of N points the data can be closely approximated by a second degree curve. It also assumes that the time interval between data samples is constant and free from error.

Each point on the smoothed data curve is obtained by fitting a second degree curve, using the least squares method, to N consecutive observation points, and evaluating the fitted curve at its midpoint.

If a second degree curve of the form

$$x = A_0 + A_1 T + A_2 T^2$$

is fitted to N data points $\{T_i, x_i\}$, then the smoothed data at any point T_i is given by

$$x_{si} = A_0 + A_1 T_i + A_2 T_i^2 \qquad (1)$$

where $i = t-N+1, t-N+2, \cdots, t - \frac{N-1}{2}, \cdots, t-2, t-1, t$.

The midpoint of this interval, T_p, occurs at the point T_i, where

$$i = t - \frac{N-1}{2} = p.$$

If equation (1) is rewritten so that its origin is translated to the midpoint, T_p, many simplifications in the curve fitting and evaluation result.

$$x_{si} = A_0 + A_1 (T_i - T_p) + A_2 (T_i - T_p)^2 \qquad (2)$$

Since the time interval between data points is constant, $T_{i+1} - T_i = \Delta T$, and it can be seen that

$$T_i - T_p = (i-p) \Delta T.$$

Equation (2) can be written

$$x_{si} = A_0 + A_1 (i-p) \Delta T + A_2 (i-p)^2 \Delta T^2 \qquad (3)$$

This is further simplified by the substitutions:

$$a_o = A_o$$

$$a_1 = A_1 \Delta T$$

$$a_2 = A_2 \Delta T^2$$

so that $x_{si} = a_o + a_1\,(i-p) + a_2\,(i-p)^2$ (4)

where $(i-p) = \frac{-(N-1)}{2}, \frac{-(N-3)}{2}, \ldots, \frac{(N-3)}{2}, \frac{(N-1)}{2}.$

Using the least squares procedure the sum of the squares of the residuals is given by

$$S = \sum_{i=t-N+1}^{i=t} (x_{si} - x_i)^2 = \sum_{i=t-N+1}^{i=t} (a_o + a_1\,(i-p) + a_2\,(i-p)^2 - x_i)^2 \quad (5)$$

where x_{si} = smoothed position

x_i = unsmoothed (observed) position

The sum S is minimized by equating its partial derivatives,

$\frac{\partial S}{\partial a_o}$, $\frac{\partial S}{\partial a_1}$, and $\frac{\partial S}{\partial a_2}$, to zero, and

solving the three resulting equations for a_o, a_1, and a_2.

These equations are:

$$\begin{aligned} a_o N \quad &+ a_1\,\Sigma(i-p) + a_2\,\Sigma(i-p)^2 = \Sigma x_i \\ a_o\,\Sigma(i-p) &+ a_1\,\Sigma(i-p)^2 + a_2\,\Sigma(i-p)^3 = \Sigma x_i\,(i-p) \\ a_o\,\Sigma(i-p)^2 &+ a_1\,\Sigma(i-p)^3 + a_2\,\Sigma(i-p)^4 = \Sigma x_i\,(i-p)^2 \end{aligned} \quad (6)$$

where all summations are over the interval from i=t-N+1 to i=t.

Since $(i-p)$ ranges from $\frac{-(N-1)}{2}$ to $\frac{(N-1)}{2}$ in steps of one, the summations in equations (6) can be found from special numerical relationships to be:

$$\Sigma(i-p) = \Sigma(i-p)^3 = 0$$

$$\Sigma(i-p)^2 = \frac{N(N^2-1)}{12}$$

$$\Sigma(i-p)^4 = \frac{N(N^2-1)}{12} \quad \frac{(3N^2-7)}{20}$$

After substituting these values, equations (6) can be written in matrix form:

$$\begin{pmatrix} N & 0 & \frac{N(N^2-1)}{12} \\ 0 & \frac{N(N^2-1)}{12} & 0 \\ \frac{N(N^2-1)}{12} & 0 & \frac{N(N^2-1)}{12} \quad \frac{(3N^2-7)}{20} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \Sigma x_i \\ \Sigma x_i \ (i-p) \\ \Sigma x_i \ (i-p)^2 \end{pmatrix} \qquad (7)$$

If Δ is the determinant of the coefficient matrix, and Δ_{ij} the cofactor of the element of the i^{th} row and j^{th} column, then the inverse of the coefficient matrix is formed from:

$$\begin{pmatrix} \frac{\Delta_{11}}{\Delta} & \frac{\Delta_{12}}{\Delta} & \frac{\Delta_{13}}{\Delta} \\ \frac{\Delta_{21}}{\Delta} & \frac{\Delta_{22}}{\Delta} & \frac{\Delta_{23}}{\Delta} \\ \frac{\Delta_{31}}{\Delta} & \frac{\Delta_{32}}{\Delta} & \frac{\Delta_{33}}{\Delta} \end{pmatrix} = \begin{pmatrix} \frac{3(3N^2-7)}{4N(N^2-4)} & 0 & \frac{-15}{N(N^2-4)} \\ 0 & \frac{12}{N(N^2-1)} & 0 \\ \frac{-15}{N(N^2-4)} & 0 & \frac{12}{N(N^2-1)} \quad \frac{15}{(N^2-4)} \end{pmatrix} \qquad (8)$$

Premultiplying equation (7) by equation (8) yields the solution in matrix form for the unknowns a_0, a_1, and a_2.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\Delta_{11}}{\Delta} & \frac{\Delta_{12}}{\Delta} & \frac{\Delta_{13}}{\Delta} \\ \frac{\Delta_{21}}{\Delta} & \frac{\Delta_{22}}{\Delta} & \frac{\Delta_{23}}{\Delta} \\ \frac{\Delta_{31}}{\Delta} & \frac{\Delta_{32}}{\Delta} & \frac{\Delta_{33}}{\Delta} \end{pmatrix} \begin{pmatrix} \Sigma x_i \\ \Sigma x_i \ (i-p) \\ \Sigma x_i \ (i-p)^2 \end{pmatrix} \tag{9}$$

or

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{3(3N^2-7)}{4N(N^2-4)} & 0 & \frac{-15}{N(N^2-4)} & \\ 0 & \frac{12}{N(N^2-1)} & 0 & \\ \frac{-15}{N(N^2-4)} & 0 & \frac{12}{N(N^2-1)} & \frac{15}{(N^2-4)} \end{pmatrix} \begin{pmatrix} \sum_{i=t-N+1}^{i=t} x_i \\ \sum_{i=t-N+1}^{i=t} x_i \ (i-p) \\ \sum_{i=t-N+1}^{i=t} x_i \ (i-p)^2 \end{pmatrix} \tag{10}$$

These values can easily be found and substituted in equation (4) to obtain the smoothed position of the arc at time T_i. The curve evaluated at its midpoint, $T_i = T_p$, is simply

$$x_{si} = a_0$$

Velocities

The velocity is obtained by taking the first derivative of the smoothed position equation (4).

$$\dot{x}_{si} = \frac{d \ x_{si}}{dT} = \frac{d}{dT} \left[a_0 + a_1 \ (i-p) + a_2 \ (i-p)^2 \right] \tag{11}$$

Since $(i-p) = \dfrac{T_i - T_p}{\Delta t}$

$$\frac{d}{dT}(i-p) = \frac{1}{\Delta T}$$

and $\dfrac{d}{dT}(i-p)^2 = \dfrac{2\,(i-p)}{\Delta T}$

Thus it can be seen that equation (11) can be written

$$\dot{x}_{s_i} = \frac{a_1}{\Delta T} + \frac{2a_2\,(i-p)}{\Delta T} \tag{12}$$

The velocity at the midpoint, i=p, is given by

$$\dot{x}_{s_i} = \frac{a_1}{\Delta T} \tag{13}$$

Acceleration

The acceleration is computed from the second derivative of the smoothed position equation, or from equation (12)

$$\ddot{x}_{s_i} = \frac{d}{dT}(\dot{x}_{r_i}) = \frac{d}{dT}\left[\frac{a_1}{\Delta T} \quad \frac{2a_2}{\Delta T}(i-p)\right] \tag{14}$$

This yields

$$\ddot{x}_{s_i} = \frac{2a_2}{\Delta T^2} \tag{15}$$

Standard Deviations

To obtain the standard deviations of the smoothed data and the derivatives, it is first necessary to find the variance of the unsmoothed data. This variance is

$$\sigma_x^2 = \frac{\Sigma(x_i - x_{s_i})^2}{N-3} \tag{16}$$

The variance of a point on the smoothed curve is obtained as follows:

If $x_{s_i} = a_o + a_1 (i-p) + s_2 (i-p)^2$

then $$\sigma^2_{x_{s_i}} = \left(\frac{\partial x_{s_i}}{\partial a_o}\right)^2 \sigma^2_{a_o} + \left(\frac{\partial x_{s_i}}{\partial a_1}\right)^2 \sigma^2_{a_1} + \left(\frac{\partial x_{s_i}}{\partial a_2}\right)^2 \sigma^2_{a_2}$$

$$+ 2\left(\frac{\partial x_{s_i}}{\partial a_o}\right)\left(\frac{\partial x_{s_i}}{\partial a_1}\right) \sigma_{a_o a_1} + 2\left(\frac{\partial x_{s_i}}{\partial a_o}\right)\left(\frac{\partial x_{s_i}}{\partial a_2}\right) \sigma_{a_o a_2}$$

$$+ 2\left(\frac{\partial x_{s_i}}{\partial a_1}\right)\left(\frac{\partial x_{s_i}}{\partial a_2}\right) \sigma_{a_1 a_2} \tag{17}$$

where $$\frac{\partial x_{s_i}}{\partial a_o} = 1$$

$$\frac{\partial x_{s_i}}{\partial a_1} = (i-p)$$

$$\frac{\partial x_{s_i}}{\partial a_2} = (i-p)^2$$

and

$$\sigma^2_{a_o} = \frac{\Delta_{11}}{\Delta} \sigma^2_x$$

$$\sigma^2_{a_1} = \frac{\Delta_{22}}{\Delta} \sigma^2_x$$

$$\sigma^2_{a_2} = \frac{\Delta_{33}}{\Delta} \sigma^2_x$$

$$\sigma^2_{a_o a_1} = 0$$

$$\sigma^2_{a_o a2} = \frac{\Delta_{13}}{\Delta} \sigma^2_x$$

$$\sigma^2_{a_1 a_2} = 0$$

By substituting these values in equation (17), the variance of x_{s_i} becomes

$$\sigma^2_{x_{si}} = \frac{\Delta_{11}}{\Delta}\,\sigma^2_x + (i-p)^2\,\frac{\Delta_{22}}{\Delta}\,\sigma^2_x + (i-p)^4\,\frac{\Delta_{33}}{\Delta}\,\sigma^2_x$$

$$+\ 2\,(i-p)^2\left(\frac{\Delta_{13}}{\Delta}\right)\sigma^2_x$$

$$= \sigma^2_x\left[\frac{\Delta_{11}}{\Delta} + (i-p)^2\left(\frac{\Delta_{22}}{\Delta} + 2\,\frac{\Delta_{13}}{\Delta}\right) + (i-p)^4\left(\frac{\Delta_{33}}{\Delta}\right)\right] \qquad (18)$$

Evaluating (18) at the point i=p gives the variance

$$\sigma^2_{x_{s_p}} = \frac{\Delta_{11}}{\Delta}\,\sigma^2_x = \frac{3(3N^2-7)}{4N(N^2-4)}\,\sigma^2_x\ .$$

or the standard deviation

$$\sigma_{x_{s_p}} = \left(\frac{\Delta_{11}}{\Delta}\,\sigma^2_x\right)^{\frac{1}{2}} \qquad (19)$$

The variance of the velocity is derived from equation (12) as follows:

$$\dot{x}_{s_i} = \frac{a_1}{\Delta T} + \frac{2a_2(i-p)}{\Delta T} \qquad (12)$$

$$\sigma^2_{\dot{x}_{s_i}} = \left(\frac{\partial \dot{x}_{s_i}}{\partial a_1}\right)^2 \sigma^2_{a_1}\ \left(\frac{\partial \dot{x}_{s_i}}{\partial a_2}\right)^2 \sigma^2_{a_2} + 2\left(\frac{\partial \dot{x}_{s_i}}{\partial a_1}\right)\left(\frac{\partial \dot{x}_{s_i}}{\partial a_2}\right)\sigma_{a_1a_2} \qquad (20)$$

where $\left(\frac{\partial \dot{x}_{s_i}}{\partial a_1}\right) = \frac{1}{\Delta T}$

$$\left(\frac{\partial \dot{x}_{s_i}}{\partial a_2}\right) = \frac{2(i-p)}{\Delta T}$$

and $\sigma^2_{a_1}$, $\sigma^2_{a_2}$ and $\sigma_{a_1a_2}$ are as in equation (17).

Substituting these values, equation (20) becomes

$$\sigma^2_{\dot{x}_{s_i}} = \frac{1}{\Delta T^2}\left(\frac{\Delta_{22}}{\Delta}\right)\sigma^2_x + \frac{4(i-p)^2}{\Delta T^2}\left(\frac{\Delta_{33}}{\Delta}\right)\cdot\sigma^2_x$$

$$= \frac{\sigma^2_x}{\Delta T^2}\left(\frac{\Delta_{22}}{\Delta} + 4\,(i-p)^2\,\frac{\Delta_{33}}{\Delta}\right) \tag{21}$$

The variance of velocity evaluated at point i=p is then

$$\sigma^2_{\dot{x}_{s_p}} = \frac{1}{\Delta T^2}\left(\frac{\Delta_{22}}{\Delta}\right)\sigma^2_x$$

$$= \left(\frac{1}{\Delta T^2}\right)\left(\frac{12}{N(N^2-1)}\right)\sigma^2_x$$

and the standard deviation

$$\sigma_{\dot{x}_{s_p}} = \left[\frac{1}{\Delta T^2}\left(\frac{12}{N(N^2-1)}\right)\sigma^2_x\right]^{\frac{1}{2}} \tag{22}$$

The variance of the acceleration is obtained from equation (15)

$$\ddot{x}_{s_i} = \frac{2\,a_2}{\Delta T^2} \tag{15}$$

$$\sigma^2_{\ddot{x}_{s_i}} = \left(\frac{d\,\ddot{x}_{s_i}}{d\,a_2}\right)^2\sigma^2_{a_2}$$

$$= \left(\frac{2}{\Delta T^2}\right)^2\left(\frac{\Delta_{33}}{\Delta}\right)\sigma^2_x$$

$$= \left(\frac{2}{\Delta T^2}\right)^2\left(\frac{12}{N(N^2-1)}\cdot\frac{15}{(N^2-4)}\right)\sigma^2_x$$

The standard deviation of the acceleration is then

$$\sigma_{\ddot{x}_{s_i}} = \sigma_{\ddot{x}_{sp}} = \left[\left(\frac{2}{\Delta T^2}\right)^2 \left(\frac{\Delta_{33}}{\Delta}\right) \sigma_x^2\right]^{\frac{1}{2}} \tag{23}$$

Advancing the Midpoint One Point

After evaluating the fitted curve on the interval t-N+1 through t to find the smoothed position, velocity and acceleration at the midpoint p, the curve may be advanced one point, to cover the interval t-N+2 through t+1, and evaluated at the new midpoint p' = p+1. The coefficient matrix of equations (9) and (10) will remain the same, but the summations about the new midpoint must be recomputed.

If the previous sums are represented by

$$A_p = \sum_{t-N+1}^{t} x_i$$

$$B_p = \sum_{t-N+1}^{t} x_i\ (i-p)$$

$$C_p = \sum_{t-N+1}^{t} x_i\ (i-p)^2$$

and the summations about the new midpoint (p') are

$$A_{p'} = \sum_{t-N+2}^{t+1} x_i$$

$$B_{p'} = \sum_{t-N+2}^{t+1} x_i\ (i-p')$$

$$C_{p'} = \sum_{t-N+2}^{t+1} x_i\ (i-p')^2$$

then $A_{p'}$, $B_{p'}$, $C_{p'}$ can be written as functions of A_p, B_p, C_p, the points to be dropped and added as follows:

$$A_{p'} = A_{p+1} = A_p - x_{t-N+1} + x_{t+1} \tag{24}$$

$$\begin{aligned} B_{p'} = B_{p+1} &= \sum_{t-N+2}^{t+1} x_i \, (i-p') \\ &= \sum_{t-N+2}^{t+1} x_i \, (i-(p+1)) \\ &= \sum_{t-N+2}^{t+1} x_i \, (i-p) - \sum_{t-N+2}^{t+1} x_i \\ &= \left[B_p - x_{t-N+1} \, (i-p) + x_{t+1} \, (i-p) \right] - \sum_{t-N+2}^{t+1} x_i \\ &= B_p - x_{t-N+1} \, (i-p) + x_{t+1} \, (i-p) - A_{p'} \end{aligned} \tag{25}$$

since $p = t - \left(\frac{N-1}{2}\right)$, when $i=t-N+1$, $(i-p) = -\left(\frac{N-1}{2}\right)$

$$\text{and when } i=t+1 \text{ , } (i-p) = \left(\frac{N+1}{2}\right) \tag{26}$$

Substituting these values in equation (25) yields

$$B_{p'} = B_p - A_{p'} + \left(\frac{N-1}{2}\right) x_{t-N+1} + \left(\frac{N+1}{2}\right) x_{t+1} \tag{27}$$

$$C_{p'} = C_{p+1} = \sum_{t-N+2}^{t+1} x_i \, (i-p')^2$$

$$= \sum_{t-N+2}^{t+1} x_i \left[(i-p)^2 - 2\,(i-p) + 1 \right]$$

$$= \sum_{t-N+2}^{t+1} x_i \, (i-p)^2 - 2 \sum_{t-N+2}^{t+1} x_i \, (i-p) + \sum_{t-N+2}^{t+1} x_i$$

$$= \left[C_p - x_{t-N+1} \, (i-p)^2 + x_{t+1} \, (i-p)^2 \right] - 2 \left[B_{p'} + \sum_{t-N+2}^{t+1} x_i \right]$$

$$+ \sum_{t-N+2}^{t+1} x_i$$

$$= C_p - 2B_{p'} - A_{p'} - x_{t-N+1} \, (i-p)^2 + x_{t+1} \, (i-p)^2 \qquad (28)$$

Substituting the relationships of equation (26) into (28) yields

$$C_{p'} = C_p - 2\,B_{p'} - A_{p'} - \left(\frac{N-1}{2}\right)^2 x_{t-N-2} + \left(\frac{N+1}{2}\right)^2 x_{t+1} \qquad (29)$$

Summarizing, the three equations are

$$A_{p'} = A_p - x_{t-N+1} + x_{t+1} \qquad (24)$$

$$B_{p'} = B_p - A_{p'} + \left(\frac{N-1}{2}\right) x_{t-N+1} + \left(\frac{N+1}{2}\right) x_{t+1} \qquad (27)$$

$$C_{p'} = C_p - 2\,B_{p'} - A_{p'} - \left(\frac{N-1}{2}\right)^2 x_{t-N+1} + \left(\frac{N+1}{2}\right)^2 x_{t+1} \qquad (29)$$

Increasing N and Advancing the Midpoint

After evaluating the fitted curve at the midpoint p of N points, it may be desirable to advance the midpoint to $p' = p+1$ not by dropping the point x_{t-N+1} and adding point x_{t+1}, but by increasing the number of points to $N' = N+2$ by adding the two points x_{t+1} and x_{t+2}. In this case the coefficient matrix must be recomputed and it can be shown that the summations about the new midpoint (p'), are functions of the previous summations and the two additional points.

Again, the previous sums are

$$A_p = \sum_{t-N+1}^{t} x_i$$

$$B_p = \sum_{t-N+1}^{t} x_i \ (i-p)$$

$$C_p = \sum_{t-N+1}^{t} x_i \ (i-p)^2$$

and the new sums are

$$A_{p'} = \sum_{t-N+1}^{t+2} x_i$$

$$B_{p'} = \sum_{t-N+1}^{t+2} x_i \ (i-p')$$

$$C_{p'} = \sum_{t-N+1}^{t+2} x_i \ (i-p')^2$$

Then: $$A_{p'} = \sum_{t-N+1}^{t+2} x_i = \sum_{t-N+1}^{t} x_i + x_{t+1} + x_{t+2}$$

$$= A_p + x_{t+1} + x_{t+2} \qquad (30)$$

$$B_{p'} = \sum_{t-N+1}^{t+2} x_i \, (i-p') = \sum_{t-N+1}^{t+2} x_i \, (i-p) - \sum_{t-N+1}^{t+2} x_i$$

$$= B_p + x_{t+1} \, (i-p) + x_{t+2} \, (i-p) - A_{p'} \tag{31}$$

Since when $N' = N+2$ and $p = t - \left(\frac{N-1}{2}\right) = t - \left(\frac{N'-3}{2}\right)$,

for $i=t+1$, $(i-p) = \left(\frac{N'-1}{2}\right)$

and for $i=t+2$, $(i-p) = \left(\frac{N'-1}{2}\right)$. (32)

Using relations (32) in equation (31),

$$B_{p'} = B_p - A_{p'} + \left(\frac{N'-1}{2}\right) x_{t+1} + \left(\frac{N'+1}{2}\right) x_{t+2} \tag{33}$$

$$C_{p'} = \sum_{t-N+1}^{t+2} x_i \, (i-p')^2 = \sum_{t-N+1}^{t+2} x_i \, (i-p)^2 - 2 \sum_{t-N+1}^{t+2} x_i \, (i-p) + \sum_{t-N+1}^{t+2} x_i$$

$$= \left[C_p + x_{t+1} \, (i-p)^2 + x_{t+2} \, (i-p)^2 \right] - 2 \left[B_{p'} + A_{p'} \right] + A_{p'}$$

$$= C_p - 2 \, B_{p'} - A_{p'} + x_{t+1} \, (i-p)^2 + x_{t+2} \, (i-p)^2 \tag{34}$$

or, substituting relations (32),

$$C_{p'} = C_p - 2 \, B_{p'} - A_{p'} + \left(\frac{N'-1}{2}\right)^2 x_{t+1} + \left(\frac{N'+1}{2}\right)^2 x_{t+2} \tag{35}$$

And summarizing

$$A_{p'} = A_p + x_{t+1} + x_{t+2} \tag{30}$$

$$B_{p'} = B_p - A_{p'} + \left(\frac{N'-1}{2}\right) x_{t+1} + \left(\frac{N'+1}{2}\right) x_{t+2} \tag{33}$$

$$C_{p'} = C_p - 2\, B_{p'} - A_{p'} + \left(\frac{N'-1}{2}\right)^2 x_{t+1} + \left(\frac{N'+1}{2}\right)^2 x_{t+2} \tag{35}$$

Decreasing N and Advancing the Midpoint

By dropping the first two points of an arc, the midpoint is advanced one point. Thus $N'=N-2$ and $p'=p+1$: Again the coefficient matrix must be recomputed and the summations about the midpoint may be written as functions of the previous sums and the two points to be subtracted, x_{t-N+1} and x_{t-N+2}.

A similar procedure is used in decreasing the interval as was used in increasing the interval yielding the following results for the summations about the new midpoint.

$$A_{p'} = A_p - x_{t-N+1} - x_{t-N+2} \tag{36}$$

$$B_{p'} = B_p - A_{p'} + \left(\frac{N'+1}{2}\right) x_{t-N+1} + \left(\frac{N'-1}{2}\right) x_{t-N+2} \tag{37}$$

$$C_{p'} = C_p - 2\, B_{p'} - A_{p'} - \left(\frac{N'+1}{2}\right)^2 x_{t-N+1} - \left(\frac{N'-1}{2}\right)^2 x_{t-N+2} \tag{38}$$

where A_p, B_p, and C_p are the previous summations.

Computational Procedure

A. Position, Velocity and Acceleration

Compute $\Delta T = T_{i+1} - T_i$

Solve for a_o, a_1, a_2 using the following matrix form

$$\begin{pmatrix} a_o \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{3(3N^2-7)}{4N(N^2-4)} & 0 & \frac{-15}{N(N^2-4)} \\ 0 & \frac{12}{N(N^2-1)} & 0 \\ \frac{-15}{N(N^2-4)} & 0 & \frac{12}{N(N^2-1)} \ \frac{15}{(N^2-4)} \end{pmatrix} \begin{pmatrix} \sum_{i=t-N+1}^{t} x_i \\ \sum_{i=t-N+1}^{t} x_i\,(i-p) \\ \sum_{i=t-N+1}^{t} x_i\,(i-p)^2 \end{pmatrix}$$

Then at the midpoint i=p the smoothed position, velocity and acceleration are:

$$x_{s_i} = a_o$$

$$\dot{x}_{s_i} = \frac{a_1}{\Delta T}$$

$$\ddot{x}_{s_i} = \frac{2a_2}{\Delta T^2}$$

B. Standard Deviation

Compute the variance of the unsmoothed data by

$$\sigma_x^2 = \frac{\Sigma(x_i - x_{si})^2}{N-3}$$

where $x_{s_i} = a_o + a_1\,(i-p) + a_2\,(i-p)^2$

$$(i-p) = \frac{-(N-1)}{2},\ \frac{-(N-3)}{2},\ \ldots,\ \frac{(N-3)}{2},\ \frac{(N-3)}{2}$$

The standard deviations of the midpoint position and derivatives are:

$$\sigma_{x_{sp}} = \left[\frac{3(3N^2-7)}{4N(N^2-4)} \right]^{1/2} \sigma_x$$

$$\sigma_{\dot{x}_{sp}} = \left[\frac{1}{\Delta T^2} \left(\frac{12}{N(N^2-1)} \right) \right]^{1/2} \sigma_x$$

$$\sigma_{\ddot{x}_{sp}} = \left[\left(\frac{2}{\Delta T^2} \right)^2 \left(\frac{12}{N(N^2-1)} \quad \frac{15}{(N^2-4)} \right) \right]^{1/2} \sigma_x$$

DETERMINATION OF THE SMOOTHING INTERVAL

This is a simple method of determining the smoothing interval (N) to be used for smoothed positions, velocities or accelerations.

When the variance and the sampling rate of the unsmoothed position data are known it is possible to choose N such that the variance of the midpoint of the smoothed positions, velocities or accelerations is equal to, or less than the required variance.

Using equations (18), (22), and (23) we see that for positions,

$$\left[\frac{3(3N^2 - 7)}{4N(N^2 - 4)}\right]^{\frac{1}{2}} = \frac{\sigma_{x_{sp}}}{\sigma_x}$$

velocities,

$$\left[\frac{12}{N(N^2 - 1)}\right]^{\frac{1}{2}} = \Delta T \frac{\sigma_{\dot{x}_{sp}}}{\sigma_x}$$

and accelerations,

$$2\left[\frac{12}{N(N^2 - 1)} \cdot \frac{15}{(N^2 - 4)}\right]^{\frac{1}{2}} = \frac{\Delta T^2 \sigma_{\ddot{x}_{sp}}}{\sigma_x}$$

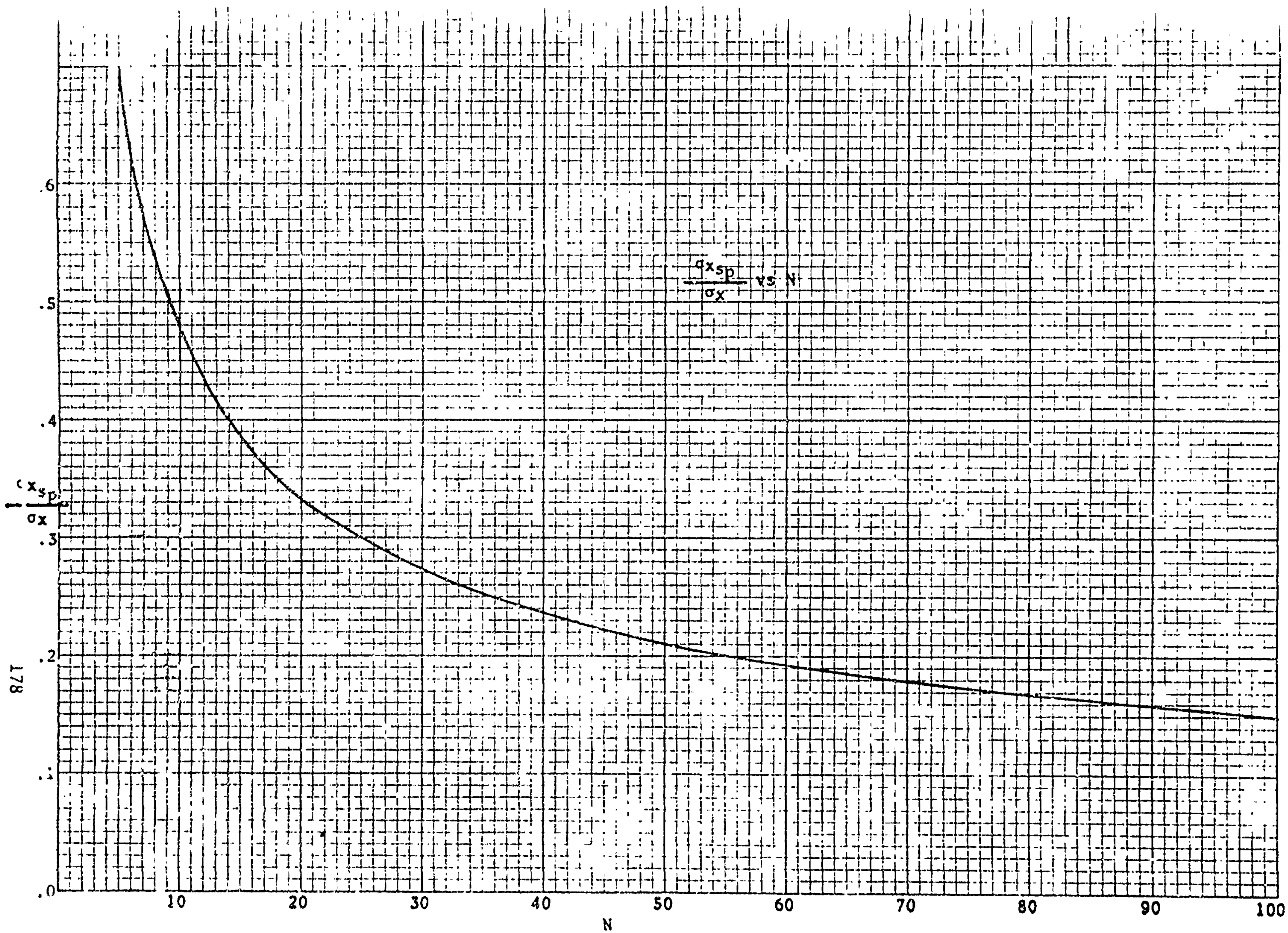
The value on the right side of each of these equations is determined from the unsmoothed position variance, σ_x, and the required variance ($\sigma_{x_{sp}}$, $\sigma_{\dot{x}_{sp}}$, or $\sigma_{\ddot{x}_{sp}}$), and the sampling rate. The only variable on the left side of each equation is N, which may then be varied to meet the desired requirements.

Table I has been prepared for this purpose and may be used in the following manner. Compute the right side of the equation for position, velocity or acceleration. Under the appropriate column of the table find the value which is equal to, or less than, the computed value and use the N associated with this value for the smoothing interval.

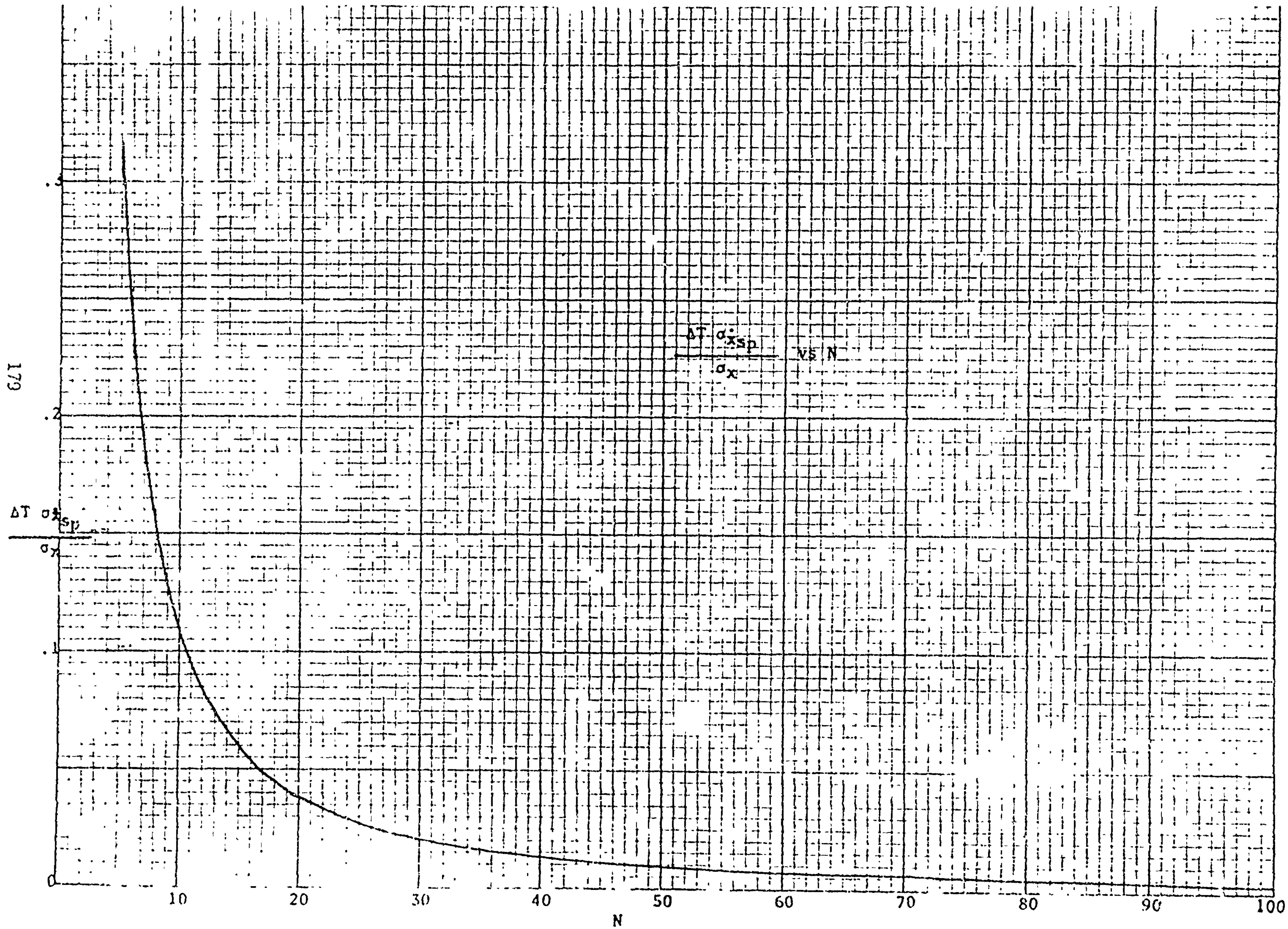
Graphs of Table I, with N extended to 100, are also provided.

TABLE 1

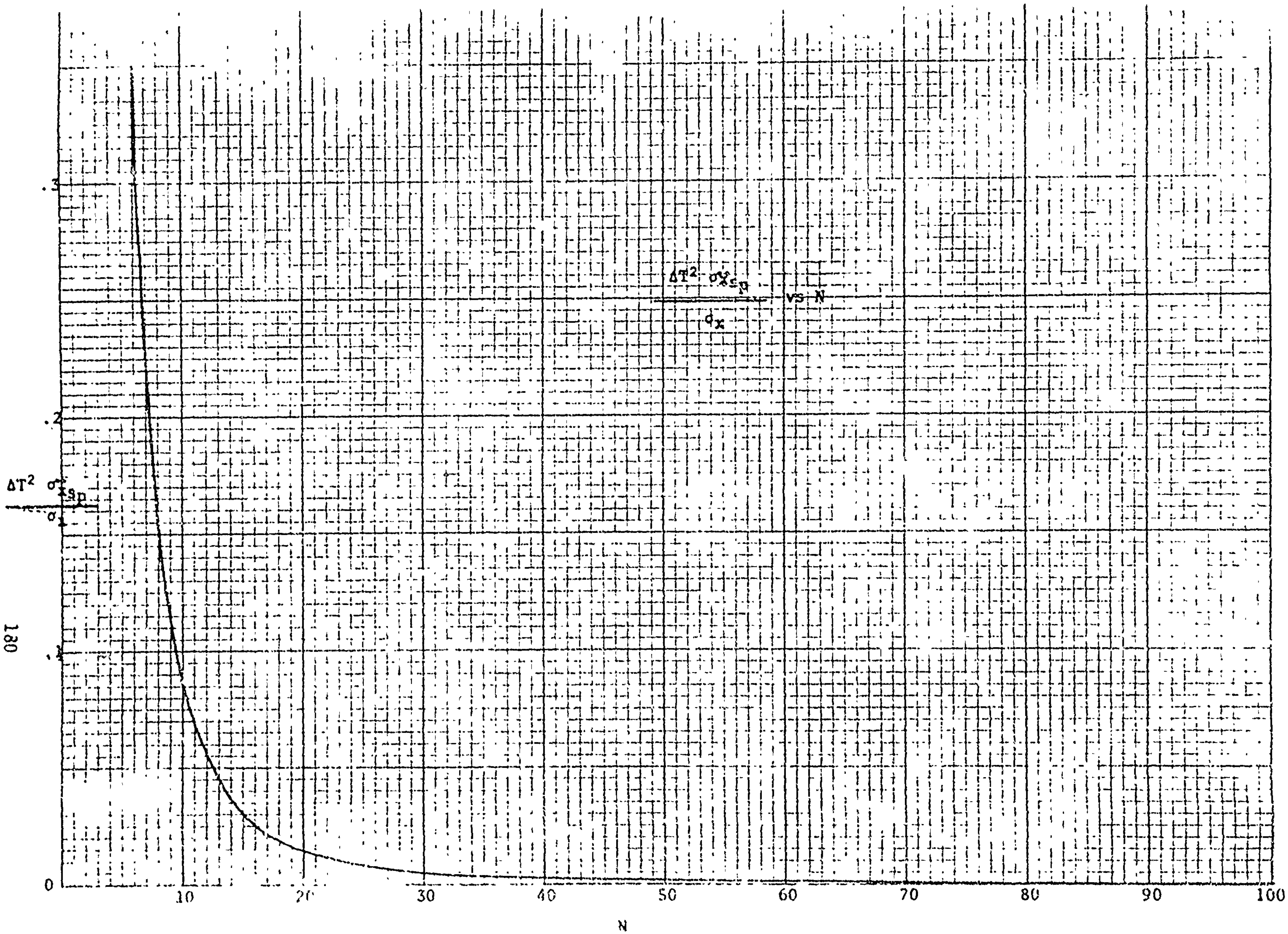
N	Position $\left[\frac{3(3N^2 - 7)}{4N(N^2 - 4)}\right]^{\frac{1}{2}}$	Velocity $\left[\frac{12}{N(N^2 - 1)}\right]^{\frac{1}{2}}$	Acceleration $2\left[\frac{12}{N(N^2 - 1)} \; \frac{15}{(N^2 - 4)}\right]^{\frac{1}{2}}$
5	.696932	.316228	.534522
7	.577350	.188982	.218218
9	.505382	.129099	.113961
11	.455477	.095346	.068279
13	.418121	.074125	.044699
15	.388756	.059761	.031139
17	.364866	.049507	.022716
19	.344926	.041885	.017171
21	.327950	.036037	.013353
23	.313268	.031435	.010627
25	.300402	.027735	.008621
27	.289007	.024708	.007108
29	.278820	.022195	.005942
31	.269642	.020080	.005028
33	.261317	.018282	.004299
35	.253719	.016737	.003710
37	.246749	.015397	.003228
39	.240324	.014228	.002830
41	.234377	.013199	.002497
43	.228851	.012289	.002216
45	.223699	.011478	.001978
47	.218880	.010753	.001774
49	.214360	.010102	.001598
51	.210109	.009513	.001446



$\frac{\sigma_{x_{sp}}}{\sigma_x}$ vs N
$\frac{\sigma_{x_{sp}}}{\sigma_x}$
.6
.5
.4
.3
.2
.1
.0
10
20
30
40
50
60
70
80
90
100
N



$\frac{\Delta T\ \sigma_{\dot{x}sp}}{\sigma_x}$ vs N
$\frac{\Delta T\ \sigma_{\dot{x}sp}}{\sigma_x}$
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
N


ΔT² σ²x_sp / σ_x vs N
ΔT² σ²_sp / σ_1
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
N

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D. VELOCITY AND ACCELERATION

II Smoothed Positions, Velocity and Acceleration (Orthogonal Polynomials), including Special Numerical Relationships

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Smoothed Positions, Velocity and Acceleration (Orthogonal Polynomials)

Introduction

In smoothing data by the usual method of least squares it is necessary to choose in advance the degree of polynomial which will be used to approximate the data. This is necessary since the coefficients found are dependent upon the degree of curve being fitted. Often, however, it is not known in advance what degree curve will best fit the data. In such a case it is desirable to fit several polynomials, each time increasing the degree used, until it is seen that any further increases would not produce a significantly better fit. The computation of successive polynomials is greatly simplified by the use of the Orthogonal Polynomial procedure. This method determines the approximating polynomial in terms of another variable, so chosen that each coefficient found is independent of the others. This makes it possible to increase the degree of curve used without making it necessary to recompute the previously-found coefficients.

This program is generally used to smooth position data. The degree of curve fitted is increased until an F-test indicates that additional coefficients of the polynomial would not be significantly different from zero. The smoothed positions are then differentiated to obtain velocities, and the velocities differentiated to obtain accelerations. The error estimates of the smoothed data and derivatives are computed in the form of standard deviations for each point. Coefficients of the original polynomial are derived in terms of the new polynomial.

Mathematical Derivation

Smoothed Position

If an approximating polynomial of the form

$$X_{s_t} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_k t^k \qquad (1)$$

is rewritten in terms of another variable which is a function of t such that the coefficients are functions of the approximating polynomial and are independent of one another, then the solution of the equation can be greatly simplified. Such an equation is of the form

$$X_{s_t} = b_0\, P_{0,t} + b_1\, P_{1,t} + b_2\, P_{2,t} + b_3\, P_{3,t} + \cdots + b_k\, P_{k,t} \qquad (2)$$

where $b_0 = f_0\,(a_0, a_1, a_2, \cdots, a_k)$

$b_1 = f_1\,(a_0, a_1, a_2, \cdots, a_k)$

$b_2 = f_2\,(a_0, a_1, a_2, \cdots, a_k)$

$\vdots$

$b_k = f_k\,(a_0, a_1, a_2, \cdots a_k)$

and $P_{0,t} = f_0\,(t)$

$P_{1,t} = f_1\,(t)$

$P_{2,t} = f_2\,(t)$

$\vdots$

$P_{k,t} = f_k\,(t)$

The following procedure derives the coefficients (b_i) such that they satisfy the above requirements for the desired polynomial.

Using the least squares method on equation (1), the sum of the squares of the residuals is given by

$$S = \Sigma(X_{s_t} - X_t)^2 = \Sigma(a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k - X_t)^2 \qquad (3)$$

where X_{s_t} = smoothed positions

X_t = observed positions

$$t = -\left(\frac{N-1}{2}\right), -\left(\frac{N-3}{2}\right), \cdots, \left(\frac{N-3}{2}\right), \left(\frac{N-1}{2}\right)$$

k = degree of the polynomial

N = number of consecutive observation points.

The sum, S, is minimized by equating the partial derivatives,

$\frac{\partial S}{\partial a_0}, \frac{\partial S}{\partial a_1}, \cdots, \frac{\partial S}{\partial a_k}$, to zero, yielding k+1 equations in k+1 unknowns.

These equations are:

$$a_0N + a_1\Sigma t + a_2\Sigma t^2 + a_3\Sigma t^3 + a_4\Sigma t^4 + a_5\Sigma t^5 + a_6\Sigma t^6 + \cdots + a_k\Sigma t^k = \Sigma X_t \tag{4}$$

$$a_0\Sigma t + a_1\Sigma t^2 + a_2\Sigma t^3 + a_3\Sigma t^4 + a_4\Sigma t^5 + a_5\Sigma t^6 + a_6\Sigma t^7 + \cdots + a_k\Sigma t^{k+1} = \Sigma X_t t$$

$$a_0\Sigma t^2 + a_1\Sigma t^3 + a_2\Sigma t^4 + a_3\Sigma t^5 + a_4\Sigma t^6 + a_5\Sigma t^7 + a_6\Sigma t^8 + \cdots + a_k\Sigma t^{k+2} = \Sigma X_t t^2$$

$$a_0\Sigma t^3 + a_1\Sigma t^4 + a_2\Sigma t^5 + a_3\Sigma t^6 + a_4\Sigma t^7 + a_5\Sigma t^8 + a_6\Sigma t^9 + \cdots + a_k\Sigma t^{k+3} = \Sigma X_t t^3$$

$$a_0\Sigma t^4 + a_1\Sigma t^5 + a_2\Sigma t^6 + a_3\Sigma t^7 + a_4\Sigma t^8 + a_5\Sigma t^9 + a_6\Sigma t^{10} + \cdots + a_k\Sigma t^{k+4} = \Sigma X_t t^4$$

$$a_0\Sigma t^5 + a_1\Sigma t^6 + a_2\Sigma t^7 + a_3\Sigma t^8 + a_4\Sigma t^9 + a_5\Sigma t^{10} + a_6\Sigma t^{11} + \cdots + a_k\Sigma t^{k+5} = \Sigma X_t t^5$$

$$\vdots$$

$$a_0\Sigma t^k + a_1\Sigma t^{k+1} + a_2\Sigma t^{k+2} + a_3\Sigma t^{k+3} + a_4\Sigma t^{k+4} + a_5\Sigma t^{k+5} + a_6\Sigma t^{k+6}$$

$$+ \cdots + a_k\Sigma t^{k+k} = \Sigma X_t t^k$$

Since t ranges from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ in steps of one, the summations in equation (4) can be found from special numerical relationships* to be:

*See page 37.

$$\Sigma t = \Sigma t^3 = \Sigma t^5 = \Sigma t^7 = 0 \tag{5}$$

$$\Sigma t^2 = \frac{N(N^2 - 1)}{12}$$

$$\Sigma t^4 = \left(\frac{3N^2 - 7}{20}\right) \Sigma t^2$$

$$\Sigma t^6 = \left(\frac{3N^4 - 18N^2 + 31}{112}\right) \Sigma t^2$$

$$\Sigma t^8 = \left(\frac{5N^6 - 55N^4 + 239N^2 - 381}{960}\right) \Sigma t^2$$

$$\Sigma t^{10} = \left(\frac{3N^8 - 52N^6 + 410N^4 - 1636N^2 + 2555}{2816}\right) \Sigma t^2$$

Using these summations, equations (4) become

$$a_0 N + a_2 \frac{N(N^2-1)}{12} + a_4 \frac{N(N^2-1)(3N^2-7)}{(12)20} + a_6 \frac{N(N^2-1)(3N^4-18N^2+31)}{(12)112} + \cdots + a_k \Sigma t^k = \Sigma X_t \tag{6a}$$

$$a_1 \frac{N(N^2-1)}{12} + a_3 \frac{N(N^2-1)(3N^2-7)}{(12)20} + a_5 \frac{N(N^2-1)(3N^4-18N^2+31)}{(12)112} + \cdots + a_k \Sigma t^{k+1} = \Sigma X_t t \tag{6b}$$

$$a_0 \frac{N(N^2-1)}{12} + a_2 \frac{N(N^2-1)(3N^2-7)}{(12)20} + a_4 \frac{N(N^2-1)(3N^4-18N^2+31)}{(12)112}$$

$$+ a_6 \frac{N(N^2-1)(5N^6-55N^4+239N^2-381)}{(12)960} + \cdots + a_k \Sigma t^{k+2} = \Sigma X_t t^2 \tag{6c}$$

$$a_1 \frac{N(N^2-1)(3N^2-7)}{(12)20} + a_3 \frac{N(N^2-1)(3N^4-18N^2+31)}{(12)112} + a_5 \frac{N(N^2-1)(5N^6-55N^4+239N^2-381)}{(12)960}$$

$$+ \cdots + a_k \Sigma t^{k+3} = \Sigma X_t t^3 \tag{6d}$$

$$a_0 \frac{N(N^2-1)(3N^2-7)}{(12)20} + a_2 \frac{N(N^2-1)(3N^4-18N^2+31)}{(12)112} + a_4 \frac{N(N^2-1)(5N^6-55N^4+239N^2-381)}{(12)960} +$$

$$a_6 \frac{N(N^2-1)(3N^8-52N^6+410N^4-1636N^2-2555)}{2816} + \cdots + a_k \Sigma t^{k+4} = \Sigma X_t t^4 \tag{6e}$$

$$a_1 \frac{N(N^2 - 1)(3N^4 - 18N^2 + 31)}{(12)112} + a_3 \frac{N(N^2 - 1)(5N^6 - 55N^4 + 239N^2 - 381)}{(12)960} +$$

$$a_5 \frac{N(N^2 - 1)(3N^8 - 52N^6 + 410N^4 - 1636N^2 + 2555)}{(12)2816} + \cdots + a_k \Sigma t^{k+5} = \Sigma X_t t^5 \tag{6f}$$

$$\vdots$$

$$a_0 \Sigma t^k + a_1 \Sigma t^{k+1} + a_2 \Sigma t^{k+2} + a_3 \Sigma t^{k+3} + a_4 \Sigma t^{k+4} + a_5 \Sigma t^{k+5} + \cdots + a_k \Sigma t^{k+k} = \Sigma X_t t^k \tag{6g}$$

From equations (6a) and (6b), for any k^{th} degree polynomial the general equations for a_0 and a_1 are

$$a_0 = \frac{\Sigma X_t}{N} - a_2 \frac{(N^2 - 1)}{12} - a_4 \frac{(N^2 - 1)(3N^2 - 7)}{(12)20} - a_6 \frac{(N^2 - 1)(3N^4 - 18N^2 + 31)}{(12)112} - \cdots - a_k \frac{\Sigma t^k}{N} \tag{7}$$

$$a_1 = \frac{\Sigma X_t t}{\frac{N(N^2 - 1)}{12}} - a_3 \frac{(3N^2 - 7)}{20} - a_5 \frac{(3N^4 - 18N^2 + 31)}{112} - \cdots - \frac{a_k \Sigma t^{k+1}}{\frac{N(N^2 - 1)}{12}} \tag{8}$$

Since the coefficients of the desired polynomial are to be independent of one another, the j^{th} coefficient will remain the same for any kth degree ($k \geq j$) polynomial. Therefore, we will use the simplest case, where $k=1$, to obtain b_0 and b_1 from equations 7 and 8.

If $k = 1$, then from equations (7) and (8),

$$a_0 = \frac{\Sigma X_t}{N} \tag{9}$$

$$a_1 = \frac{\Sigma X t}{\frac{N(N^2 - 1)}{12}} \tag{10}$$

Since a_0 and a_1 are independent of one another and are functions of the approximating polynomial, they satisfy the requirements for the coefficients of the desired polynomial. Therefore

let $$b_0 = \frac{\Sigma X_t}{N} \tag{11}$$

and $$b_1 = \frac{\Sigma X t}{\frac{N(N^2 - 1)}{12}} \tag{12}$$

To find the next two coefficients, b_2 and b_3, we must first substitute the general expressions for a_0 and a_1 (eq. 7 and 8) into equations 6(c thru g).

$$a_2 \frac{N(N^2 - 1)(N^2 - 4)}{(12)15} + a_4 \frac{N(N^2 - 1)(N^2 - 4)(3N^2 - 13)}{(12)(15)(14)} + a_6 \frac{N(N^2 - 1)(N^2 - 4)(5N^4 - 50N^2 + 157)}{(12)(15)(112)}$$

$$+ \cdots + a_k \left[\Sigma t^{k+2} - \frac{(N^2 - 1)}{12} \Sigma t^k\right] = \Sigma X_t t^2 - \frac{(N^2 - 1)}{12} \Sigma X_t \tag{13c}$$

$$a_3 \frac{N(N^2 - 1)(N^2 - 4)(N^2 - 9)9}{(12)(15)(140)} + a_5 \frac{N(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 7)}{(12)(15)(8)(7)} + \cdots +$$

$$a_k \left[\Sigma t^{k+3} - \frac{(3N^2 - 7)}{20} \Sigma t^{k+1}\right] = \Sigma X_t t^3 - \frac{(3N^2 - 7)}{20} \Sigma X_t t \tag{13d}$$

$$a_2 \frac{N(N^2 - 1)(N^2 - 4)(3N^2 - 13)}{(12)(15)(14)} + a_4 \frac{N(N^2 - 1)(N^2 - 4)(N^4 - 10N^2 + 29)}{(12)(15)(20)} +$$

$$a_6 \frac{N(N^2 - 1)(N^2 - 4)(27N^6 - 459N^4 + 3141N^2 - 8309)}{(12)(15)(11)(14)(16)} + \cdots + a_k \left[\Sigma t^{k+4} - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)} \Sigma t^k\right] =$$

$$\Sigma X_t t^4 - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)} \Sigma X_t \tag{13e}$$

$$a_3 \frac{N(N^2-1)(N^2-4)(N^2-7)(N^2-9)}{(12)(15)(14)(4)} + a_5 \frac{N(N^2-1)}{12} \frac{(3N^8 - 85N^6 + 905N^4 - 4343N^2 + 7164)}{(16)(11)(49)}$$

$$+ \cdots + a_k \left[\Sigma t^{k+5} - \frac{(3N^4 - 18N^2 + 31)}{112} \Sigma t^{k+1} \right] = \Sigma X_t t^5 - \frac{(3N^4 - 18N^2 + 31)}{112} \Sigma X_t \tag{13f}$$

From equations (13c) and (13d) the general equations for a_2 and a_3 are found to be

$$a_2 = \frac{\Sigma X_t t^2 - \frac{(N^2-1)}{12} \Sigma X_t}{\frac{N(N^2-1)(N^2-4)}{(12)(15)}} - \frac{a_4 (3N^2 - 13)}{14} - \frac{a_6 (5N^4 - 50N^2 + 157)}{112} - \cdots - \frac{a_k \left[\Sigma t^{k+2} - \frac{(N^2-1)}{12} \Sigma t^k \right]}{\frac{N(N^2-1)(N^2-4)}{(12)(15)}} \tag{14}$$

$$a_3 = \frac{\Sigma X_t t^3 - \frac{(3N^2-7)}{20} \Sigma X_t t}{\frac{N(N^2-1)(N^2-4)\ 9(N^2-9)}{(12)(15)(140)}} - \frac{a_5 (N^2-7)\ 5}{(9)(2)} - \cdots - \frac{a_k \left[\Sigma t^{k+3} - \frac{(3N^2-7)}{20} \Sigma t^{k+1} \right]}{\frac{N(N^2-1)(N^2-4)\ 9(N^2-9)}{(12)(15)(140)}} \tag{15}$$

Again, using the lowest degree polynomial, in this case k=3, and solving for a_2 and a_3 in equations (14) and (15),

$$a_2 = \frac{\Sigma X_t t^2 - \frac{(N^2-1)}{12} \Sigma X_t}{\frac{N(N^2-1)(N^2-4)}{(12)(15)}} \tag{16}$$

$$a_3 = \frac{\Sigma X_t t^3 - \frac{(3N^2-7)}{20} \Sigma X_t t}{\frac{N(N^2-1)(N^2-4)(9)(N^2-9)}{(12)(15)(140)}} \tag{17}$$

Since equations (16) and (17) satisfy the requirements of the new coefficients, we let

$$b_2 = \frac{\Sigma X_t t^2 - \frac{(N^2 - 1)}{12} \Sigma X_t}{\frac{N(N^2 - 1)(N^2 - 4)}{(12)(15)}} \tag{18}$$

$$\text{and } b_3 = \frac{\Sigma X_t t^3 - \frac{(3N^2 - 7)}{20} \Sigma X_t t}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)}{(12)(15)(140)}} \tag{19}$$

In a similar way, the general equations for a_4 and a_5 are obtained, after substituting equations (14) and (15) into equation (13g) and (13f).

$$a_4 = \frac{\left[\Sigma X_t t - \frac{(N^2 - 1)}{12} \frac{(3N^2 - 7)}{20} \Sigma X_t - \frac{(3N^2 - 13)}{14} \left(\Sigma X_t t^2 - \frac{(N^2 - 1)}{12} \Sigma X_t\right)\right]}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)}{(12)(15)(140)(63)(4)(99)}} - \frac{a_6\ 5(3N^2 - 31)}{(11)(4)}$$

$$- \cdots - \frac{a_k \left[\Sigma t^{k+4} - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)} \Sigma t^k - \frac{(3N^2 - 31)}{14} \left(\Sigma t^{k+2} - \frac{(N^2 - 1)}{12} \Sigma t^k\right)\right]}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)}{(12)(15)(140)(63)(4)(99)}} \tag{20}$$

$$a_5 = \frac{\Sigma X_t t^5 - \frac{(3N^4 - 18N^2 + 31)}{112} \Sigma X_t t - \frac{5(N^2 - 7)}{18}\left(\Sigma X_t t^3 - \frac{(3N^2 - 7)}{20} \Sigma X_t t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)(25)(N^2 - 25)}{(12)(15)(140)(63)(4)(99)}} - \cdots$$

$$- \frac{a_k \left[\Sigma t^{k+5} - \frac{(3N^4 - 18N^2 + 31)}{112} \Sigma t^{k+1} - \frac{5(N^2 - 7)}{18}\left(\Sigma t^{k+3} - \frac{(3N^2 - 7)}{20}\right)\Sigma t^{k+1}\right]}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)\ 25(N^2 - 25)}{(12)(15)(140)(63)(4)(99)}} \tag{21}$$

Using the lowest degree polynomial for this case, k=5, then

$$a_4 = \frac{\Sigma X_t t^4 - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)} \Sigma X_t - \frac{(3N^2 - 13)}{14}\left(\Sigma X_t t^2 - \frac{(N^2 - 1)}{12} \Sigma X_t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)}{(12)(15)(140)(63)}} \tag{22}$$

$$a_5 = \frac{\Sigma X_t t^5 - \frac{(3N^4 - 18N^2 + 31)}{112} \Sigma X_t t - \frac{5(N^2 - 7)}{18}\left(\Sigma X_t t^3 - \frac{(3N^2 - 7)}{20} \Sigma X_t t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)\ 25(N^2 - 25)}{(12)(15)(140)(63)(4)(99)}} \tag{23}$$

These two equations again satisfy the required conditions for the new coefficients and

$$b_4 = \frac{\Sigma X_t t^4 - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)} \Sigma X_t - \frac{(3N^2 - 13)}{14}\left(\Sigma X_t t^2 - \frac{(N^2 - 1)}{12} \Sigma X_t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)}{(12)(15)(140)(63)}} \tag{24}$$

$$b_5 = \frac{\Sigma X_t t^5 - \frac{(3N^4 - 18N^2 + 31)}{112}\Sigma X_t t - \frac{5(N^2 - 7)}{18}\left(\Sigma X_t t^3 - \frac{(3N^2 - 7)}{20}\Sigma X_t t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\,9(N^2 - 9)\,4(N^2 - 16)\,25(N^2 - 25)}{(12)(15)(140)(63)(4)(99)}} \tag{25}$$

This procedure could be continued until the k^{th} coefficient is determined, but this is a long and tedious method. An easier method, based on a recursion relationship, exists and is derived below. Summarizing the b_j's we have

$$b_0 = \frac{\Sigma X_t}{N}$$

$$b_1 = \frac{\Sigma X_t t}{\frac{N(N^2 - 1)}{12}}$$

$$b_2 = \frac{\Sigma X_t t^2 - \frac{(N^2 - 1)}{12}\Sigma X_t}{\frac{N(N^2 - 1)(N^2 - 4)}{(12)(15)}} \tag{26}$$

$$b_3 = \frac{\Sigma X_t t^3 - \frac{(3N^2 - 7)}{20}\Sigma X_t t}{\frac{N(N^2 - 1)(N^2 - 4)\,9(N^2 - 9)}{(12)(15)(140)}}$$

$$b_4 = \frac{\Sigma X_t t^4 - \frac{(N^2 - 1)(3N^2 - 7)}{(12)(20)}\Sigma X_t - \frac{(N^2 - 13)}{14}\left(\Sigma X_t t^2 - \frac{(N^2 - 1)}{12}\Sigma X_t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\,9(N^2 - 9)\,4(N^2 - 16)}{(12)(15)(140)(63)}}$$

$$b_5 = \frac{\Sigma X_t^5 - \frac{(3N^4 - 18N^2 + 31)}{112}\,\Sigma X_t - \frac{5(N^2 - 7)}{18}\left(\Sigma X_t^3 - \frac{(3N^2 - 7)}{20}\,\Sigma X_t\right)}{\frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)\ 25(N^2 - 25)}{(12)(15)(140)(63)(4)(99)}}$$

If we let the denominators of the b_j's be represented by Q_j's, then

$$Q_0 = N$$

$$Q_1 = \frac{N(N^2 - 1)}{12} = Q_0\,\frac{(N^2 - 1)}{12} = \frac{Q_0\,(N^2 - 1^2)}{(4 \cdot 3)} = \frac{Q_0\,(N^2 - 1^2)}{4[4(1^2) - 1]}$$

$$Q_2 = \frac{N(N^2 - 1)(N^2 - 4)}{(12)(15)} = Q_1\,\frac{(N^2 - 4)}{15} = \frac{Q_1\,(N^2 - 2^2)}{4(4) - 1} = \frac{Q_1\,4(N^2 - 2^2)}{4[4(2^2) - 1]} = \frac{Q_1\,(2^2)(N^2 - 2^2)}{4[4(2^2) - 1]} \tag{27}$$

$$Q_3 = \frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)}{(12)(15)(140)} = \frac{Q_2\,(9)(N^2 - 9)}{140} = \frac{Q_2\,(9)(N^2 - 3^2)}{4(35)} = \frac{Q_2\,(9)(N^2 - 3^2)}{4[4(9) - 1]} = \frac{Q_2\,(3^2)(N^2 - 3^2)}{4[4(3^2) - 1]}$$

$$Q_4 = \frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)\ 4(N^2 - 16)}{(12)(15)(140)(63)} = \frac{Q_3\,4(N^2 - 16)}{63} = \frac{Q_3\,(4)(N^2 - 4^2)}{[4(16) - 1]} = \frac{Q_3\,(4)^2\,(N^2 - 4^2)}{4[4(4^2) - 1]}$$

$$Q_5 = \frac{N(N^2 - 1)(N^2 - 4)(9)(N^2 - 9)(4)(N^2 - 16)(25)(N^2 - 25)}{(12)(15)(140)(63)(4)(99)} = \frac{Q_4\,(25)(N^2 - 25)}{4(99)} = \frac{Q_4\,(5^2)(N^2 - 5^2)}{4[100 - 1]}$$

$$= \frac{Q_4\,(5^2)(N^2 - 5^2)}{4[4(5^2) - 1]}$$

It is easily seen that the recursion formula for the j^{th} denominator is

$$Q_j = Q_{j-1}\left(\frac{j^2(N^2 - j^2)}{4[4(j)^2 - 1]}\right)$$

Substituting the Q_j's in equations (26) and combining terms in the numerators

$$b_0 = \frac{\Sigma X_t}{Q_0}$$

$$b_1 = \frac{\Sigma X_t}{Q_1}$$

$$b_2 = \frac{\Sigma X_t\left[t^2 - \frac{(N^2 - 1)}{12}\right]}{Q_2}$$

$$b_3 = \frac{\Sigma X_t\left[t^3 - t\,\frac{(3N^2 - 7)}{20}\right]}{Q_3} \tag{28}$$

$$b_4 = \frac{\Sigma X_t\left[t^4 - t^2\,\frac{(3N^2 - 13)}{14} + \frac{3(N^2 - 1)(N^2 - 9)}{560}\right]}{Q_4}$$

$$b_5 = \frac{\Sigma X_t\left[t^5 - t^3\left(\frac{5(N^2 - 7)}{18}\right) + t\left(\frac{15N^4 - 230N^2 + 407}{(16)(63)}\right)\right]}{Q_5}$$

The polynomials in the numerators on the right sides of equations (28) are obviously functions of t and will be represented by $P_{j,t}$.

$$
\begin{aligned}
P_{0,t} &= 1 \\
P_{1,t} &= t \\
P_{2,t} &= t^2 - \frac{(N^2 - 1)}{12} \\
P_{3,t} &= t^3 - \frac{t(3N^2 - 7)}{20} \\
P_{4,t} &= t^4 - t^2 \frac{(3N^2 - 13)}{14} + \frac{3(N^2 - 1)(N^2 - 9)}{560} \\
P_{5,t} &= t^5 - t^3 \frac{5(N^2 - 7)}{18} + t \frac{(15N^4 - 230N^2 + 407)}{(4)(4)(63)}
\end{aligned}
\qquad (29)
$$

In order to derive a recursion relationship for the $P_{j,t}$'s, we rewrite equation 29 in terms of the Q_j's of equation 27:

$$P_{0,t} = 1$$

$$P_{1,t} = t$$

$$P_{2,t} = t^2 - \frac{Q_1}{Q_0} = t(t) - 1\left(\frac{Q_1}{Q_0}\right)$$

$$P_{3,t} = t\left[t^2 - \left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1}\right)\right] = t\left[t^2 - \frac{Q_1}{Q_0}\right] - t\left(\frac{Q_2}{Q_1}\right) \tag{30}$$

$$P_{4,t} = t\left[t^3 - t\,\frac{(3N^2 - 7)}{20}\right] - \frac{9(N^2 - 9)}{140}\left[t^2 - \frac{(N^2 - 1)}{12}\right] = t\left[t^3 - t\left(\frac{Q_2}{Q_1} + \frac{Q_1}{Q_0}\right)\right] - \frac{Q_3}{Q_2}\left[t^2 - \left(\frac{Q_2}{Q_1}\right)\right]$$

$$P_{5,t} = t^5 - t^3\left[\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1} + \frac{Q_3}{Q_2} + \frac{Q_4}{Q_3}\right] + t\left[\left(\frac{Q_1}{Q_0}\right)\left(\frac{Q_3}{Q_2}\right) + \left(\frac{Q_1}{Q_0}\right)\left(\frac{Q_4}{Q_3}\right) + \left(\frac{Q_2}{Q_1}\right)\left(\frac{Q_4}{Q_3}\right)\right]$$

$$= t\left[t^4 - t^2\left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1} + \frac{Q_3}{Q_2}\right) + \left(\frac{Q_1}{Q_0}\right)\left(\frac{Q_3}{Q_2}\right)\right] - \frac{Q_4}{Q_3}\left[t^3 - t\left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1}\right)\right]$$

It can easily be shown from equation (30) that the following recursion relationship exists for the $P_{j,t}$'s:

$$
\begin{aligned}
P_{0,t} &= 1 \\
P_{1,t} &= t \\
P_{2,t} &= t(t) - 1\left(\frac{Q_1}{Q_0}\right) = t(P_{1,t}) - \frac{Q_1}{Q_0}(P_{0,t}) \\
P_{3,t} &= t\left(t^2 - \frac{Q_1}{Q_0}\right) - t\left(\frac{Q_2}{Q_1}\right) = t(P_{2,t}) - \frac{Q_2}{Q_1}(P_{1,t}) \\
P_{4,t} &= t\left[t^3 - t\left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1}\right)\right] - \frac{Q_3}{Q_2}\left(t^2 - \frac{Q_1}{Q_0}\right) = t(P_{3,t}) - \frac{Q_3}{Q_2}(P_{2,t}) \\
P_{5,t} &= t\left[t^4 - t^2\left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1} + \frac{Q_3}{Q_2}\right) + \left(\frac{Q_1}{Q_0}\right)\left(\frac{Q_3}{Q_2}\right)\right] - \frac{Q_4}{Q_3}\left[t^3 - t\left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1}\right)\right] = t(P_{4,t}) - \frac{Q_4}{Q_3}(P_{3,t}) \\
&\ \ \vdots \\
P_{j,t} &= t(P_{j-1,t}) - \frac{Q_{j-1}}{Q_{j-2}}(P_{j-2,t})
\end{aligned}
\tag{31}
$$

From equations (27) and (29)

$$
\begin{aligned}
Q_0 &= N = \Sigma (P_{0,t})^2 \\
Q_1 &= \frac{N(N^2 - 1)}{12} = \Sigma t^2 = \Sigma (P_{1,t})^2 \\
Q_2 &= \frac{N(N^2 - 1)(N^2 - 4)}{(12)(15)} = \Sigma \left[t - \frac{(N^2 - 1)}{12} \right]^2 = \Sigma (P_{2,t})^2 \\
Q_3 &= \frac{N(N^2 - 1)(N^2 - 4)\ 9(N^2 - 9)}{(12)(15)(140)} = \Sigma \left[t^3 - t\,\frac{(3N^2 - 7)}{20} \right]^2 = \Sigma (P_{3,t})^2 \\
&\ \ \vdots \\
Q_j &= \Sigma (P_{j,t})^2
\end{aligned}
\tag{32}
$$

Substituting equations (31) and (32) in equation (28) yields the final forms of the coefficients (b_j) to be used:

$$b_0 = \frac{\Sigma X_t}{Q_0} = \frac{\Sigma X_t (P_{0,t})}{\Sigma (P_{0,t})^2}$$

$$b_1 = \frac{\Sigma X_t t}{Q_1} = \frac{\Sigma X_t (P_{1,t})}{\Sigma (P_{1,t})^2}$$

$$b_2 = \frac{\Sigma X_t \left[t\,(P_{1,t}) - \frac{Q_1}{Q_0}(P_{0,t}) \right]}{Q_2} = \frac{\Sigma X_t (P_{2,t})}{\Sigma (P_{2,t})^2}$$

$$b_3 = \frac{\Sigma X_r\left[t\,(P_{2,t}) - \frac{Q_2}{Q_1}\,(P_{1,t})\right]}{Q_3} = \frac{\Sigma X_t(P_{3,t})}{\Sigma(P_{3,t})^2}$$

$$b_4 = \frac{\Sigma X_r\left[t\,(P_{3,t}) - \frac{Q_3}{Q_2}\,(P_{2,t})\right]}{Q_4} = \frac{\Sigma X_t(P_{4,t})}{\Sigma(P_{4,t})^2} \tag{33}$$

$$b_5 = \frac{\Sigma X_r\left[t\,(P_{4,t}) - \frac{Q_4}{Q_3}\,(P_{3,t})\right]}{Q_5} = \frac{\Sigma X_t(P_{5,t})}{\Sigma(P_{5,t})^2}$$

$$\vdots$$

$$b_j = \frac{\Sigma X_r\left[t\,(P_{j-1,t}) - \frac{Q_{j-1}}{Q_{j-2}}\,(P_{j-2,t})\right]}{Q_j} = \frac{\Sigma X_t(P_{j,t})}{\Sigma(P_{j,t})^2}$$

Now that we have expressions for the coefficients and variables of the desired polynomial a least squares procedure is applied to equation (2), in order to show that the $P_{j,t}$'s are orthogonal polynomials*. The sum of the squares of the residuals is given by

$$S = \Sigma(X_{s_t} - X_t)^2 = \Sigma(b_0\,P_{0,t} + b_1\,P_{1,t} + b_2\,P_{2,t} + \cdots + b_k\,P_{k,t} - X_t)^2 \tag{34}$$

The sum, S, is minimized by equating the partial derivatives $\frac{\partial S}{\partial b_0}, \frac{\partial S}{\partial b_1}, \frac{\partial S}{\partial b_2}, \ldots, \frac{\partial S}{\partial b_k}$ to zero, yielding the following equations:

*The $P_{j,t}$'s are by definition orthogonal polynomials if it can be shown that $\Sigma(P_{j,t}\,P_{k,t}) = 0$ for all $j \neq k$.

$$b_0\ \Sigma(P_{0,t})^2 + b_1\ \Sigma(P_{0,t}\ P_{1,t}) + b_2\ \Sigma(P_{0,t}\ P_{2,t}) + \cdots + b_k\ \Sigma(P_{0,t}\ P_{k,t}) = \Sigma X_t\ (P_{0,t}) \tag{35a}$$

$$b_0\ \Sigma(P_{0,t}\ P_{1,t}) + b_1\ \Sigma(P_{1,t})^2 + b_2\ \Sigma(P_{1,t}\ P_{2,t}) + \cdots + b_k\ \Sigma(P_{1,t}\ P_{k,t}) = \Sigma X_t(P_{1,t}) \tag{35b}$$

$$b_0\ \Sigma(P_{0,t}\ P_{2,t}) + b_1\ \Sigma(P_{1,t}\ P_{2,t}) + b_2\ \Sigma(P_{2,t})^2 + \cdots + b_k\ \Sigma(P_{2,t}\ P_{k,t}) = \Sigma X_t(P_{2,t}) \tag{35c}$$

$$\vdots$$

$$b_0\ \Sigma(P_{0,t}\ P_{k,t}) + b_1\ \Sigma(P_{1,t}\ P_{k,t}) + b_2\ \Sigma(P_{2,t}\ P_{k,t}) + \cdots + b_k\ \Sigma(P_{k,t})^2 = \Sigma X_t(P_{k,t}) \tag{35d}$$

From equation (33)

$$b_0 = \frac{\Sigma X_t(P_{0,t})}{\Sigma(P_{0,t})^2}$$

Substituting this in equation (35a) yields

$$b_1\ \Sigma(P_{0,t}\ P_{1,t}) + b_2\ \Sigma(P_{0,t}\ P_{2,t}) + \cdots + b_k\ \Sigma(P_{0,t}\ P_{k,t}) = 0 \tag{36}$$

Since the b_j's are not necessarily zero, equation (36) evaluated for $k = 1$ yields

$$\Sigma(P_{0,t}\ P_{1,t}) = 0 \tag{37}$$

Substituting equation (37) into (36) and evaluating for $k = 2$ yields

$$\Sigma(P_{0,t}\ P_{2,t}) = 0$$

By repeating this process for all k's it can be shown that

$$\Sigma(P_{0,t}\ P_{k,t}) = 0$$

From equation (33) we again see that

$$b_1 = \frac{\Sigma X_t(P_{1,t})}{\Sigma(P_{1,t})^2}$$

Substituting this in equation (35b) yields

$$b_0\ \Sigma(P_{0,t}\ P_{1,t}) + b_2\ \Sigma(P_{1,t}\ P_{2,t}) + \cdots + b_k\ \Sigma(P_{1,t}\ P_{k,t}) = 0$$

Since from equation 37 we know that

$$\Sigma(P_{0,t}\ P_{1,t}) = 0$$

Then if $k = 2$

$$\Sigma(P_{1,t}\ P_{2,t}) = 0$$

Again evaluating for each successive k it can be shown that

$$\Sigma(P_{1,t}\ P_{k,t}) = 0$$

For all b_j's substituted in equation (35) it is easily seen that

$$\Sigma(P_{j,t}\ P_{k,t}) = 0, \quad j \neq k \tag{38}$$

Since $\Sigma(P_{j,t}\ P_{k,t}) = 0$, $j \neq k$, then by definition the $P_{j,t}$'s are orthogonal polynomials.

Degree of Polynomial

To determine the degree of the polynomial that gives the best fit we must first compute an estimate (σ^2) of the variance of the raw data, and for each j^{th} coefficient an estimate ($\hat{\sigma}^2$) of the variance of the data in terms of the variance of the coefficients. We wish to test the hypothesis that the true value of the j^{th} coefficient is zero, so that a $k < j$ degree polynomial will adequately describe the data. Assuming that the true value of the j^{th} coefficient is zero, we use the F-test to determine whether the difference between the two variance estimates (σ^2 and $\hat{\sigma}^2$) is significant. If it is not, we conclude that the hypothesis is true, and set the j^{th} coefficient equal to its true value ($b_j = 0$). When two consecutive coefficients (b_{j-1} and b_j) have been set equal to zero by this process, we assume that the degree of the best fitting polynomial is determined by the last non-zero coefficient, $k = j-2$.

The estimated variance of the raw data is the sum of the squares of the residuals divided by the degrees of freedom, the degrees of freedom being the number of observations minus one minus the degree of the curve.

$$\sigma^2 = \frac{\Sigma(X_{s_t} - X_t)^2}{D.F.} \tag{39}$$

$$D.F. = (N - 1 - k)$$

The variance of the coefficients is given by the general equation

$$\sigma^2_{b_j} = \sigma^2 \frac{\Delta_{jj}}{\Delta} \tag{40}$$

where Δ is the determinant of the coefficient matrix, and Δ_{jj} is the cofactor of the element of the j^{th} row and j^{th} column. It can easily be shown, using equation (35) and the fact that the $P_{j,t}$'s are orthogonal polynomials, that

$$\frac{\Delta_{jj}}{\Delta} = \frac{1}{Q_j} \quad \text{where the } Q_j\text{'s are as given in equations (27).} \tag{41}$$

Therefore

$$\sigma^2_{b_j} = \frac{\sigma^2}{Q_j} \tag{42}$$

The b_j's and $P_{j,t}$'s found in equations (31) and (33) are substituted in equation (2) to obtain the smoothed positions and the sum of the squares of the residuals is computed.

$$\Sigma(X_{s_t} - X_t)^2 = \Sigma(b_0 P_{0,t} + b_1 P_{1,t} + b_2 P_{2,t} + \cdots + b_j P_{j,t} - X_t)^2 \tag{43}$$

$$= \Sigma X_t^2 + b_0^2 \Sigma(P_{0,t})^2 + b_1^2 \Sigma(P_{1,t})^2 + b_2^2 \Sigma(P_{2,t})^2 + \cdots + b_j^2 \Sigma(P_{j,t})^2$$

$$- 2b_0 \Sigma(P_{0,t} X_t) - 2b_1 \Sigma(P_{1,t} X_t) - 2b_2 \Sigma(P_{2,t} X_t) - \cdots$$

$$- 2b_j \Sigma(P_{j,t} X_t)$$

As was shown previously in equation (33)

$$b_j = \frac{\Sigma(P_{j,t} X_t)}{\Sigma(P_{j,t})^2}$$

so that $\Sigma(P_{j,t} X_t) = b_j \Sigma(P_{j,t})^2$

Substituting this value in equation (43) yields

$$\Sigma(X_{s_t} - X_t)^2 = \Sigma X_t^2 - b_0^2 \Sigma(P_{0,t})^2 - b_1^2 \Sigma(P_{1,t})^2 - b_2^2 \Sigma(P_{2,t})^2 - \cdots$$

$$- b_j^2 \Sigma(P_{j,t})^2 \tag{44}$$

Using equation (39) the estimated variance is found to be $\sigma^2 = \dfrac{\Sigma(X_{s_t} - X_t)^2}{D.F.}$

Another estimated variance of the raw data is found in terms of the j^{th} coefficient. From equation (42)

$$\hat{\sigma}^2 = Q_j \sigma_{b_j}^2 \tag{45}$$

where the variance of the coefficient is given by

$$\sigma_{b_j}^2 = \frac{(b_j - B_j)^2}{D.F.}$$

b_j = computed coefficient

B_j = true coefficient

D.F. = Degrees of freedom = 1

Equation (45) then becomes

$$\hat{\sigma}^2 = Q_j \frac{(b_j - B_j)^2}{1} \tag{46}$$

Now we want to test if the hypothesis, $B_j = 0$, is true. If in equation (46) we let $B_j = 0$, then

$$\hat{\sigma}^2 = Q_j\, b_j^2 \tag{47}$$

σ^2 and $\hat{\sigma}^2$ are both sample variances. To test whether these two variances are estimates of the same population variance the F-test is used.

If

$$\frac{\hat{\sigma}^2}{\sigma^2} \leq F \tag{48}$$

then the hypothesis is accepted as being true and b_j is set equal to zero. The next coefficient is tested in the same manner. When two consecutive coefficients are set equal to zero the degree of the polynomial is determined from the last coefficient which was significant. Thus

$$k = j-2$$

Velocity

The velocity is obtained by taking the first derivative of the smoothed position equation (2)

$$X_{st} = b_0\, P_{0,t} + b_1\, P_{1,t} + b_2\, P_{2,t} + \cdots + b_k\, P_{k,t} \tag{2}$$

$$\dot{X}_{st} = \frac{d\, X_{st}}{dt} = \frac{d}{dt} [b_0\, P_{0,t} + b_1\, P_{1,t} + b_2\, P_{2,t} + \cdots + b_k\, P_{k,t}]$$

$$\dot{X}_{st} = \frac{1}{\Delta t} [b_0\, \dot{P}_{0,t} + b_1\, \dot{P}_{1,t} + b_2\, \dot{P}_{2,t} + \cdots + b_k\, \dot{P}_{k,t}] \tag{49}$$

The first derivatives of equation (31) are

$$\begin{aligned}
\dot{P}_{0,t} &= 0 \\
\dot{P}_{1,t} &= 1 \\
\dot{P}_{2,t} &= 2t \\
\dot{P}_{3,t} &= 3t^2 - \left(\frac{Q_1}{Q_0} + \frac{Q_2}{Q_1}\right) \\
&\vdots \\
\dot{P}_{k,t} &= P_{k-1,t} + t\,\dot{P}_{k-1,t} - \frac{Q_{k-1}}{Q_{k-2}}\,\dot{P}_{k-1,t}
\end{aligned} \tag{50}$$

Acceleration

The acceleration is computed from the second derivative of the smoothed position equation, or from the first derivative of equation (49).

$$\ddot{X}_{st} = \frac{1}{(\Delta t)^2}\left[b_0\,\ddot{P}_{0,t} + b_1\,\ddot{P}_{1,t} + b_2\,\ddot{P}_{2,t} + \cdots + b_k\,\ddot{P}_{k,t}\right] \tag{51}$$

The derivatives of equation (50) are

$$\begin{aligned}
\ddot{P}_{0,t} &= 0 \\
\ddot{P}_{1,t} &= 0 \\
\ddot{P}_{2,t} &= 2 \\
\ddot{P}_{3,t} &= 6t \\
&\vdots \\
\ddot{P}_{k,t} &= 2\dot{P}_{k-1,t} + t\,\ddot{P}_{k-1,t} - \frac{Q_{k-1}}{Q_{k-2}}\,\ddot{P}_{k-2,t}
\end{aligned} \tag{52}$$

Standard Deviations

The standard deviation of the smoothed data is found as follows:

$$X_{st} = b_0 P_{0,t} + b_1 P_{1,t} + b_2 P_{2,t} + \cdots + b_k P_{k,t} \tag{2}$$

The variance is

$$\sigma^2_{X_{st}} = \left(\frac{\partial X_{st}}{\partial b_0}\right)^2 \sigma^2_{b_0} + \left(\frac{\partial X_{st}}{\partial b_1}\right)^2 \sigma^2_{b_1} + \left(\frac{\partial X_{st}}{\partial b_2}\right)^2 \sigma^2_{b_2} + \cdots + \left(\frac{\partial X_{st}}{\partial b_k}\right)^2 \sigma^2_{b_k} \tag{53}$$

where $\dfrac{\partial X_{st}}{\partial b_0} = P_{0,t}$

$$\frac{\partial X_{st}}{\partial b_1} = P_{1,t} \tag{54}$$

$$\frac{\partial X_{st}}{\partial b_2} = P_{2,t}$$

$$\vdots$$

$$\frac{\partial X_{st}}{\partial b_k} = P_{k,t}$$

From equation (42)

$$\sigma^2_{b_j} = \frac{\sigma^2}{Q_j}$$

Substituting these relationships into equation (53),

$$\sigma^2_{X_{st}} = (P_{0,t})^2 \frac{\sigma^2}{Q_0} + (P_{1,t})^2 \frac{\sigma^2}{Q_1} + (P_{2,t})^2 \frac{\sigma^2}{Q_2} + \cdots + (P_{k,t})^2 \frac{\sigma^2}{Q_k} \tag{55}$$

where $\sigma^2_{X_{st}}$ = variance of the smoothed data

σ^2 = variance of the observed data

The standard deviation is

$$\sigma_{X_{St}} = \sigma \left[\sum_{j=0}^{k} \frac{(P_{j,t})^2}{Q_j} \right]^{\frac{1}{2}} \tag{56}$$

The standard deviation of the velocity is obtained as follows:
From equation (49) the velocity is

$$\dot{X}_{St} = \frac{1}{\Delta t} [b_0 \dot{P}_{0,t} + b_1 \dot{P}_{1,t} + b_2 \dot{P}_{2,t} + \cdots + b_k \dot{P}_{k,t}] \tag{49}$$

The variance is

$$\sigma^2_{\dot{X}_{St}} = \left(\frac{\partial \dot{X}_{St}}{\partial b_0}\right)^2 \sigma^2_{b0} + \left(\frac{\partial \dot{X}_{St}}{\partial b_1}\right)^2 \sigma^2_{b1} + \left(\frac{\partial \dot{X}_{St}}{\partial b_2}\right)^2 \sigma^2_{b2} + \cdots + \left(\frac{\partial \dot{X}_{St}}{\partial b_k}\right)^2 \sigma^2_{bk} \tag{57}$$

where

$$\frac{\partial \dot{X}_{St}}{\partial b_0} = \frac{\dot{P}_{0,t}}{\Delta t}$$

$$\frac{\partial \dot{X}_{St}}{\partial b_1} = \frac{\dot{P}_{1,t}}{\Delta t}$$

$$\frac{\partial \dot{X}_{St}}{\partial b_2} = \frac{\dot{P}_{2,t}}{\Delta t}$$

$$\vdots$$

$$\frac{\partial \dot{X}_{St}}{\partial b_k} = \frac{\dot{P}_{k,t}}{\Delta t}$$

and

$$\sigma^2_{b_j} = \frac{\sigma^2}{Q_j}$$

Using these values the variance of the velocity becomes

$$\sigma^2_{\dot{X}_{s_t}} = \frac{\sigma^2}{\Delta t^2}\,\frac{(\dot{P}_{0,t})^2}{Q_0} + \frac{\sigma^2}{\Delta t^2}\,\frac{(\dot{P}_{1,t})^2}{Q_1} + \frac{\sigma^2}{\Delta t^2}\,\frac{(\dot{P}_{2,t})^2}{Q_2} + \cdots \tag{58}$$

$$+ \frac{\sigma^2}{\Delta t^2}\,\frac{(\dot{P}_{k,t})^2}{Q_k}$$

and the standard deviation of the velocity is

$$\sigma_{\dot{X}_{s_t}} = \frac{\sigma}{\Delta t}\left[\sum_{j=0}^{k} \frac{(\dot{P}_{j,t})^2}{Q_j}\right]^{1/2} \tag{59}$$

The standard deviation of the acceleration is found by the same method as that of the velocity.

From equation (51) the acceleration is

$$\ddot{X} = \frac{1}{\Delta t^2}\left[b_0\,\ddot{P}_{0,t} + b_1\,\ddot{P}_{1,t} + b_2\,\ddot{P}_{2,t} + b_3\,\ddot{P}_{3,t} + \cdots + b_k\,\ddot{P}_{k,t}\right] \tag{51}$$

The variance of the acceleration becomes

$$\sigma^2_{\ddot{X}_{s_t}} = \left(\frac{\partial \ddot{X}_{s_t}}{\partial b_0}\right)^2 \sigma^2_{b_0} + \left(\frac{\partial \ddot{X}_{s_t}}{\partial b_1}\right)^2 \sigma^2_{b_1} + \left(\frac{\partial \ddot{X}_{s_t}}{\partial b_2}\right)^2 \sigma^2_{b_2} + \cdots + \left(\frac{\partial \ddot{X}_{s_t}}{\partial b_k}\right)^2 \sigma^2_{b_k} \tag{60}$$

where

$$\frac{\partial \ddot{X}_{s_t}}{\partial b_0} = \frac{\ddot{P}_{0,t}}{(\Delta t)^2}$$

$$\frac{\partial \ddot{X}_{s_t}}{\partial b_1} = \frac{\ddot{P}_{1,t}}{(\Delta t)^2}$$

$$\frac{\partial \ddot{X}_{s_t}}{\partial b_2} = \frac{\ddot{P}_{2,t}}{(\Delta t)^2}$$

$$\vdots$$

$$\frac{\partial \ddot{X}_{s_t}}{\partial b_k} = \frac{\ddot{P}_{k,t}}{(\Delta t)^2}$$

and $\sigma_{b_j}^2 = \dfrac{\sigma^2}{Q_j}$

Substituting these values in equation (60) yields

$$\sigma^2_{\ddot{X}_{s_t}} = \frac{\sigma^2}{(\Delta t)^4} \frac{(\ddot{P}_{0,t})^2}{Q_0} + \frac{\sigma^2}{(\Delta t)^4} \frac{(\ddot{P}_{1,t})^2}{Q_1} + \frac{\sigma^2}{(\Delta t)^4} \frac{(\ddot{P}_{2,t})^2}{Q_2} + \cdots$$

$$+ \frac{\sigma^2}{(\Delta t)^4} \frac{(\ddot{P}_{k,t})^2}{Q_k} \tag{61}$$

The standard deviation is

$$\sigma_{\ddot{X}_{s_t}} = \frac{\sigma}{(\Delta t)^2} \left[\sum_{j=0}^{k} \frac{(\ddot{P}_{j,t})^2}{Q_j} \right]^{1/2} \tag{62}$$

<u>Original coefficients in terms of the new polynomial.</u>

If it is desirable to obtain the coefficients of the approximating polynomial in terms of the new regression polynomial this may be done by equating equations (1) and (2) and solving for the coefficients at the midpoint. Since t ranges from $-\left(\dfrac{N-1}{2}\right)$ to $\left(\dfrac{N-1}{2}\right)$ the midpoint value of t will be zero. Evaluating equation (1) and its derivatives at the midpoint we find

$$\begin{aligned}
X_{s_0} &= a_0 \\
\dot{X}_{s_0} &= a_1 \\
\ddot{X}_{s_0} &= 2!\, a_2 \\
\dddot{X}_{s_0} &= 3!\, a_3 \\
&\vdots \\
\overset{i}{X}_{s_0} &= i!\, a_i
\end{aligned} \tag{64}$$

Equation (2) and its derivatives, at the midpoint (t=0), are as follows:

$$
\begin{aligned}
X_{s0} &= b_0 + b_2 P_{2,0} + b_4 P_{4,0} + b_6 P_{6,0} + \cdots + b_k P_{k,0} \\
\dot{X}_{s0} &= b_1 \dot{P}_{1,0} + b_3 \dot{P}_{3,0} + b_5 \dot{P}_{5,0} + \cdots + b_k \dot{P}_{k,0} \qquad (65)\\
\ddot{X}_{s0} &= b_2 \ddot{P}_{2,0} + b_4 \ddot{P}_{4,0} + b_6 \ddot{P}_{6,0} + \cdots + b_k \ddot{P}_{k,0} \\
&\vdots \\
\overset{i}{X}_{s0} &= b_j \overset{i}{P}_{j,0} + b_{j+2} \overset{i}{P}_{j+2,0} + b_{j+4} \overset{i}{P}_{j+4,0} + \cdots + b_k \overset{i}{P}_{k,0}
\end{aligned}
$$

Taking the i^{th} derivatives of the j^{th} polynomials of equation (31) yields

$$
\begin{aligned}
\dot{P}_{j,t} &= P_{j-1,t} + (t)\, \dot{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right) \dot{P}_{j-2,t} \\
\ddot{P}_{j,t} &= 2\dot{P}_{j-1,t} + (t)\, \ddot{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right) \ddot{P}_{j-2,t} \qquad (66)\\
\dddot{P}_{j,t} &= 3\ddot{P}_{j-1,t} + (t)\, \dddot{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right) \dddot{P}_{j-2,t} \\
&\vdots
\end{aligned}
$$

The general equation is

$$\overset{i}{P}_{j,t} = i \left[\overset{(i-1)}{P}_{j-1,t} \right] + (t)\, \overset{i}{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right) \overset{i}{P}_{j-2,t} \qquad (67)$$

Substituting equations (64) and (66) into equation (65) it is easily seen that

$$a_i = \sum_{j=i}^{j=k} \left(\frac{\overset{i}{P}_{j,0}}{i!} \right) b_j \qquad (68)$$

This equation may also be written as

$$a_i = \sum_{j=i}^{j=k} M_{i,j}\, b_j \tag{69}$$

where $$M_{i,j} = \frac{P^{i}_{j,0}}{i!} \tag{70}$$

Using equations (67) and (70) the following recursion equation is obtained:

$$M_{i,j} = M_{i-1,j-1} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right) M_{i,j-2} \tag{71}$$

COMPUTATIONAL PROCEDURE

In computing the b_j's using equation (33), the X's are usually all of the same sign and the $P_{j,t}$'s change signs j times. In forming the sums a large positive or negative value is accumulated as t varies, then after the $P_{j,t}$'s change signs the accumulation approaches zero and may or may not pass through zero. This cycle is repeated until the final answer is obtained. It is easily seen that the large accumulative values are far greater than the final sum. Therefore in programming this problem on a single precision high speed computer it is necessary to use double precision to obtain precise answers. These large sums may be avoided and single precision used if the X_t's are replaced with a residual $(\delta_{j-1,t})$. It can be shown in the following manner that using this residual does not change the value of b_j. The residual is

$$\delta_{j-1,t} = X_t - b_0 P_{0,t} - b_1 P_{1,t} - \cdots - b_{j-1} P_{j-1,t}$$

Then

$$\Sigma\delta_{j-1,t} P_{j,t} = \Sigma P_{j,t} (X_t - b_0 P_{0,t} - b_1 P_{1,t} - \cdots - b_{j-1} P_{j-1,t})$$

$$\Sigma\delta_{j-1,t} P_{j,t} = \Sigma X_t P_{j,t} - b_0 \Sigma P_{0,t} P_{j,t} - b_1 \Sigma P_{1,t} P_{j,t} - \cdots$$

$$- b_{j-1} \Sigma P_{j-1,t} P_{j,t}$$

Since

$$\Sigma P_{j,t} P_{k,t} = 0 \text{ for } j \neq k$$

$$\Sigma\delta_{j-1,t} P_{j,t} = \Sigma X_t P_{j,t}$$

Substituting this in equation (33) yields

$$b_j = \frac{\Sigma\delta_{j-1,t} P_{j,t}}{Q_j}$$

The trend of the signs of the residuals will, in general, be the same as the orthogonal polynomial associated with the next significant b coefficient and thus the large accumulations encountered using the X_t's are avoided.

The computer program is written to fit up to a k_{max} degree curve to the data.

The first step is to compute from $j=1$ through $j=k_{max}$

$$Q_0 = N$$

$$Q_j = Q_{j-1}\left[\frac{j^2\,(N^2 - j^2)}{4(4j^2 - 1)}\right]$$

where $11 \le N \le 251$ is the number of points in the smoothing interval.

Then compute from $t = -\left(\frac{N-1}{2}\right)$ through $t = \left(\frac{N-1}{2}\right)$

$$P_{1,t} = t$$

$$P_{2,t} = t^2 - \left(\frac{Q_1}{Q_0}\right)$$

$$P_{j,t} = t\,P_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right)P_{j-2,t} \quad \text{for } j=3 \text{ thru } j=k_{max}$$

$$\dot{P}_{2,t} = 2t$$

$$\dot{P}_{3,t} = P_{2,t} + 2t^2 - \left(\frac{Q_2}{Q_1}\right)$$

$$\dot{P}_{j,t} = P_{j-1,t} + t\,\dot{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right)\dot{P}_{j-2,t} \quad \text{for } j=4 \text{ thru } j=k_{max}$$

$$\ddot{P}_{3,t} = 6t$$

$$\ddot{P}_{4,t} = 2\dot{P}_{3,t} + 6t^2 - 2\left(\frac{Q_3}{Q_2}\right)$$

$$\ddot{P}_{j,t} = 2\dot{P}_{j,t} + t\,\ddot{P}_{j-1,t} - \left(\frac{Q_{j-1}}{Q_{j-2}}\right)\ddot{P}_{j-2,t} \quad \text{for } j=5 \text{ thru } j=k_{max}$$

For each set of N data points compute

$$b_0 = \frac{\Sigma X_t}{Q_0}$$

$$\delta_t = X_t - b_0$$

$$s^2 = \Sigma\ \delta_t^2$$

The following steps are computed as j varies from 1 thru k_{max} for each set of N data points.

$$b_j = \frac{\Sigma\ P_{j,t}\ \delta_t}{Q_j}$$

From equation (44) we see that

$$\Sigma(X_{s_t} - X_t)^2 = \Sigma\delta_{j,t}^2 = \Sigma\delta_{j-1,t}^2 - b_j^2\ Q_j$$

Therefore the estimated variance from the residuals is computed using

$$\sigma^2 = \frac{s^2 - Q_j\ b_j^2}{D.F.}$$

where D.F. = N-1-j

and the estimated variance from the coefficient is computed using

$$\hat{\sigma}^2 = Q_j\ b_j^2$$

Now the j^{th} coefficient is tested for significance using the F values associated with one degree of freedom for the numerator and N-1-j degrees of freedom for the denominator.

If $\frac{\hat{\sigma}^2}{\sigma^2} > F$, then b_j is significant

and δ_t is recomputed from

$$\delta_t = \delta_t - b_j\ P_{j,t}$$

$$s^2 = \Sigma\delta_t^2$$

and if $j \neq k_{max}$, then j is increased by one and the next coefficient is computed.

If $j = k_{max}$ the degree of the curve is k_{max} and smoothed positions are then computed. $(k = k_{max})$

If $\frac{\hat{\sigma}^2}{\sigma^2} \leq F$, then b_j is not significant and is set equal to zero.

If $j \neq k_{max}$ and the previous coefficient is not equal to zero then j is increased by one and the next coefficient is computed using the new residual.

If $j \neq k_{max}$ and the previous coefficient is equal to zero then the degree of the curve is (j-2) and smoothed positions are then computed $(k = j-2)$.

If $j = k_{max}$ and the previous coefficient is equal to zero the degree of the curve is equal to (j-2) and smoothed positions are computed. $(k = j-2)$

If $j = k_{max}$ and the previous coefficient is not zero the degree of the curve becomes j-1 and smoothed positions are computed. This logic is more clearly explained by Figure 1.

The smoothed positions are computed as follows:

$$X_{s_t} = b_0 + b_1 P_{1,t} + b_2 P_{2,t} + \cdots + b_k P_{k,t}$$

or

$$X_{s_t} = X_t - \delta_t$$

The velocity is computed next from

$$\dot{X}_{s_t} = \frac{1}{\Delta t} [b_1 + b_2 \dot{P}_{2,t} + b_3 \dot{P}_{3,t} + \cdots + b_k \dot{P}_{k,t}]$$

and the acceleration is computed from

$$\ddot{X}_{s_t} = \frac{1}{\Delta t^2} [2b_2 + b_3 \ddot{P}_{3,t} + b_4 \ddot{P}_{4,t} + \cdots + b_k \ddot{P}_{k,t}]$$

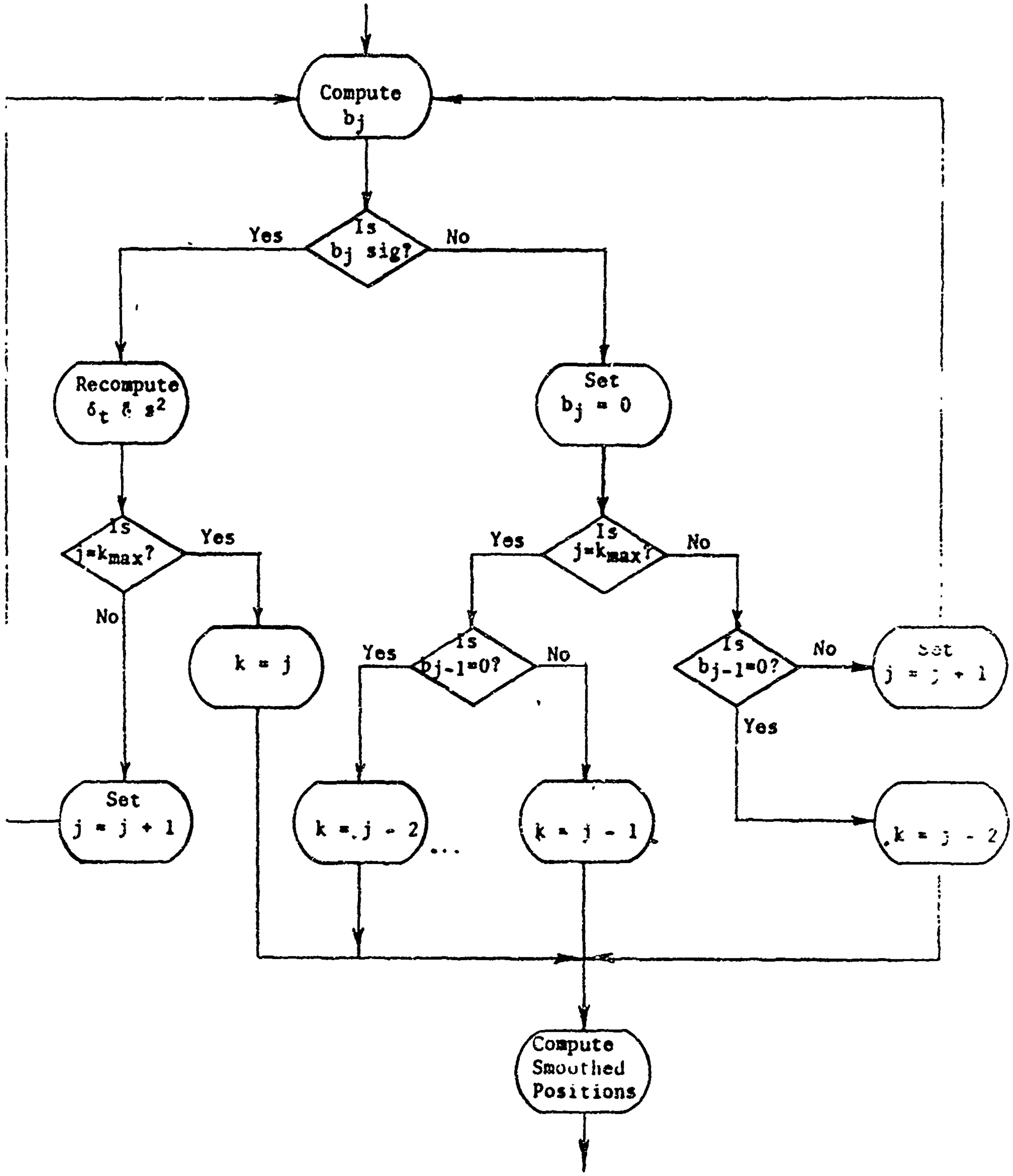
The standard deviation of the position data is computed as follows.

The variance of the observed data is computed from

$$\sigma^2 = \frac{s^2}{D.F.}$$

and the standard deviation of the smoothed position is

$$\sigma_{X_{s_t}} = \sigma \left[\frac{1}{Q_0} + \frac{(P_{1,t})^2}{Q_1} + \frac{(P_{2,t})^2}{Q_2} + \cdots + \frac{(P_{k,t})^2}{Q_k} \right]^{\frac{1}{2}}$$



Compute b_j
Is b_j sig?
Yes
No
Recompute δ_t & s^2
Set $b_j = 0$
Is $j=k_{max}$?
Yes
No
Is $j=k_{max}$?
Yes
No
$k = j$
Is $b_{j-1}=0$?
Yes
No
Is $b_{j-1}=0$?
No
Yes
Set $j = j + 1$
Set $j = j + 1$
$k = j - 2$
...
$k = j - 1$
$k = j - 2$
Compute Smoothed Positions

FIGURE 1.

Standard deviation of velocity is

$$\sigma \dot{X}_{s_t} = \frac{\sigma}{\Delta t}\left[\frac{1}{Q_1} + \frac{(\dot{P}_{2,t})^2}{Q_2} + \frac{(\dot{P}_{3,t})^2}{Q_3} + \cdots + \frac{(\dot{P}_{k,t})^2}{Q_k}\right]^{\frac{1}{2}}$$

Standard deviation of acceleration is

$$\sigma \ddot{X}_{s_t} = \frac{\sigma}{\Delta t^2}\left[\frac{4}{Q_2} + \frac{(\ddot{P}_{3,t})^2}{Q_3} + \frac{(\ddot{P}_{4,t})^2}{Q_4} + \cdots + \frac{\ddot{P}_{k,t}}{Q_k}\right]^{\frac{1}{2}}$$

Special Numerical Relationships

It is often desirable to obtain sums or sums of powers of integers whose values range from $-\left(\frac{N-1}{2}\right)$, $-\left(\frac{N-3}{2}\right)$, $\cdots$, $\left(\frac{N-3}{2}\right)$, $\left(\frac{N-1}{2}\right)$.

For the range $t = -\left(\frac{N-1}{2}\right), -\left(\frac{N-3}{2}\right), \cdots, \left(\frac{N-3}{2}\right), \left(\frac{N-1}{2}\right)$, the sum of the t's is obtained as follows:

$$\Sigma(t+1)^2 - \Sigma t^2 = \left(\frac{N-3}{2}\right)^2 + \cdots + \left(\frac{N-3}{2}\right)^2 + \left(\frac{N-1}{2}\right)^2 + \left(\frac{N+1}{2}\right)^2$$

$$- \left(\frac{N-1}{2}\right)^2 - \left(\frac{N-3}{2}\right)^2 - \cdots - \left(\frac{N-3}{2}\right)^2 - \left(\frac{N-1}{2}\right)^2$$

$$\Sigma t^2 + 2\Sigma t + N - \Sigma t^2 = \left(\frac{N+1}{2}\right)^2 - \left(\frac{N-1}{2}\right)^2$$

$$2\Sigma t + N = \frac{N^2 + 2N + 1 - N^2 + 2N - 1}{2^2}$$

$$2\Sigma t = \frac{4N}{4} - N$$

$$\Sigma t = \frac{1}{2}(N - N) = 0$$

Σt^2 is obtained from:

$$\Sigma(t+1)^3 - \Sigma t^3 = \left(\frac{N-1}{2}\right)^3 + \left(\frac{N-3}{2}\right)^3 + \cdots - \left(\frac{N-3}{2}\right)^3 - \left(\frac{N-1}{2}\right)^3$$

$$- \left(\frac{N-3}{2}\right)^3 - \cdots + \left(\frac{N-3}{2}\right)^3 + \left(\frac{N-1}{2}\right)^3 + \left(\frac{N+1}{2}\right)^3$$

$$\Sigma(t+1)^3 - \Sigma t^3 = \left(\frac{N-1}{2}\right)^3 + \left(\frac{N+1}{2}\right)^3$$

$$\Sigma t^3 + 3\Sigma t^2 + 3\Sigma t + N - \Sigma t^3 = \left(\frac{N-1}{2}\right)^3 + \left(\frac{N+1}{2}\right)^3$$

Since $\Sigma t = 0$

$$3\Sigma t^2 + N = \frac{2N^3 + 6N}{2^3} = \frac{N^3 + 3N}{2^2}$$

$$\Sigma t^2 = \frac{1}{3}\left[\frac{N^3 + 3N - 4N}{2^2}\right]$$

$$\Sigma t^2 = \frac{N(N^2 - 1)}{12}$$

For Σt^3:

$$\Sigma(t+1)^4 - \Sigma t^4 = \left(\frac{N-3}{2}\right)^4 + \cdots + \left(\frac{N-3}{2}\right)^4 + \left(\frac{N-1}{2}\right)^4 + \left(\frac{N+1}{2}\right)^4$$

$$- \left(\frac{N-1}{2}\right)^4 - \left(\frac{N-3}{2}\right)^4 - \cdots - \left(\frac{N-3}{2}\right)^4 - \left(\frac{N-1}{2}\right)^4$$

$$\Sigma t^4 + 4\Sigma t^3 + 6\Sigma t^2 + 4\Sigma t + N - \Sigma t^4 = \left(\frac{N+1}{2}\right)^4 - \left(\frac{N-1}{2}\right)^4$$

Since $\quad \Sigma t = 0$ and $\Sigma t^2 = \dfrac{N(N^2 - 1)}{12}$

$$4\Sigma t^3 + 6\Sigma t^2 + N = \frac{N^4 + 4N^3 + 6N^2 + 4N + 1 - N^4 + 4N^3 - 6N^2 + 4N - 1}{2^4}$$

$$4\Sigma t^3 = \frac{8N^3 + 8N}{2^4} - \frac{6N(N^2 - 1)}{12} - N$$

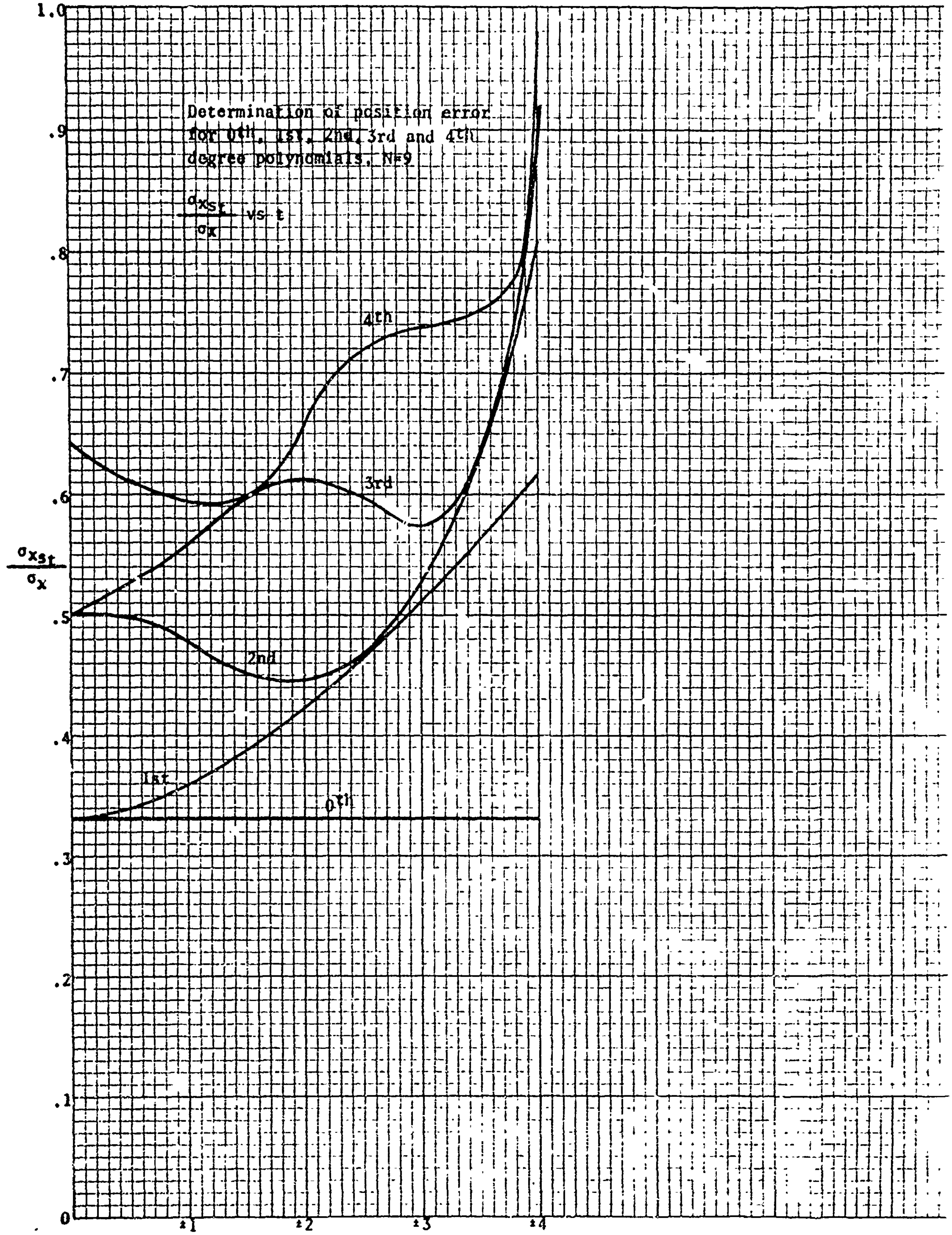
$$\Sigma t^3 = \frac{1}{4}\left[\frac{N^3 + N}{2} - \left(\frac{N^3 - N}{2}\right) - N\right]$$

$$\Sigma t^3 = \frac{1}{4}\left[\frac{2N - 2N}{2}\right]$$

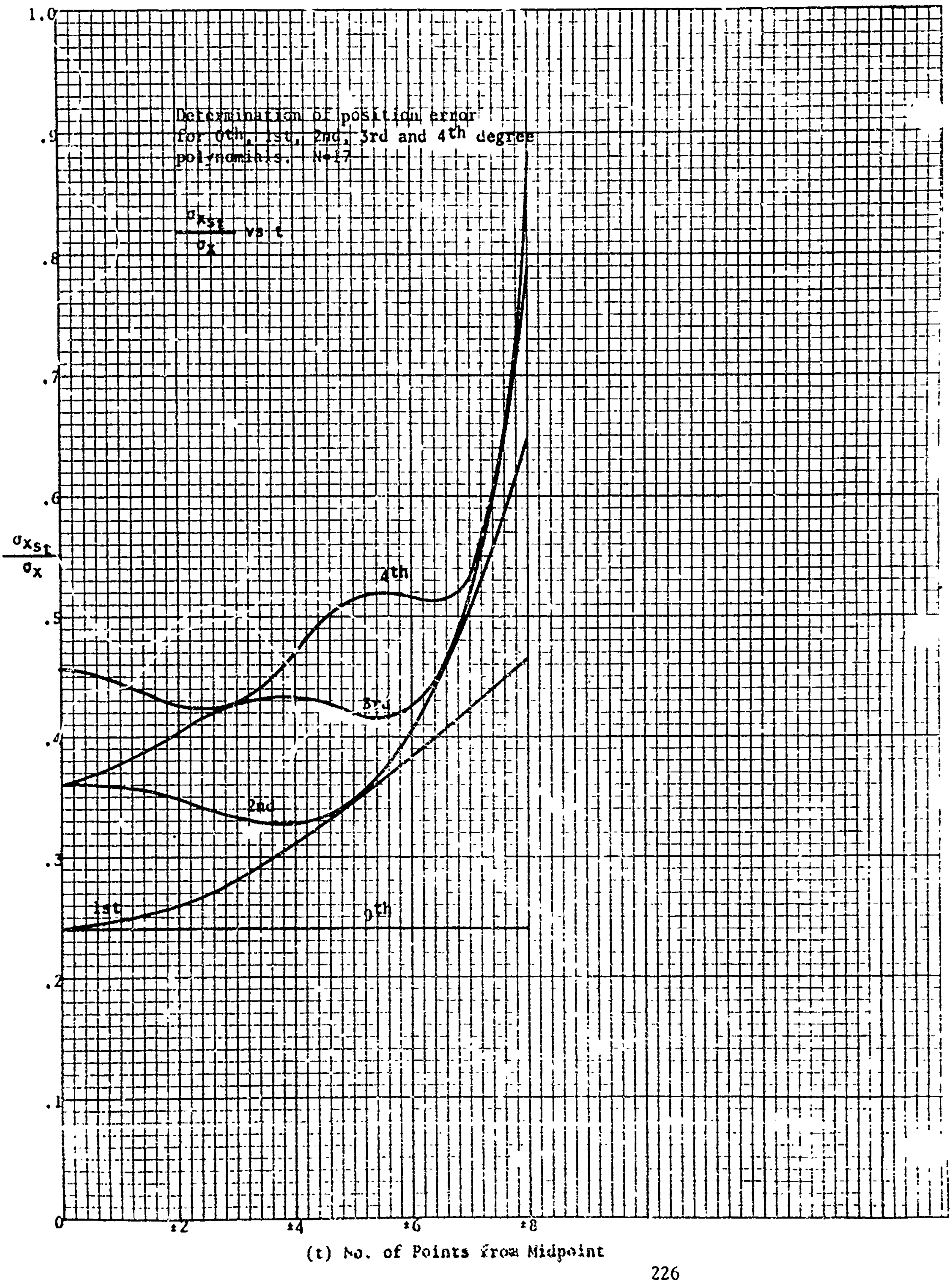
$$\Sigma t^3 = 0$$

The summations of any power can be obtained in a similar manner.

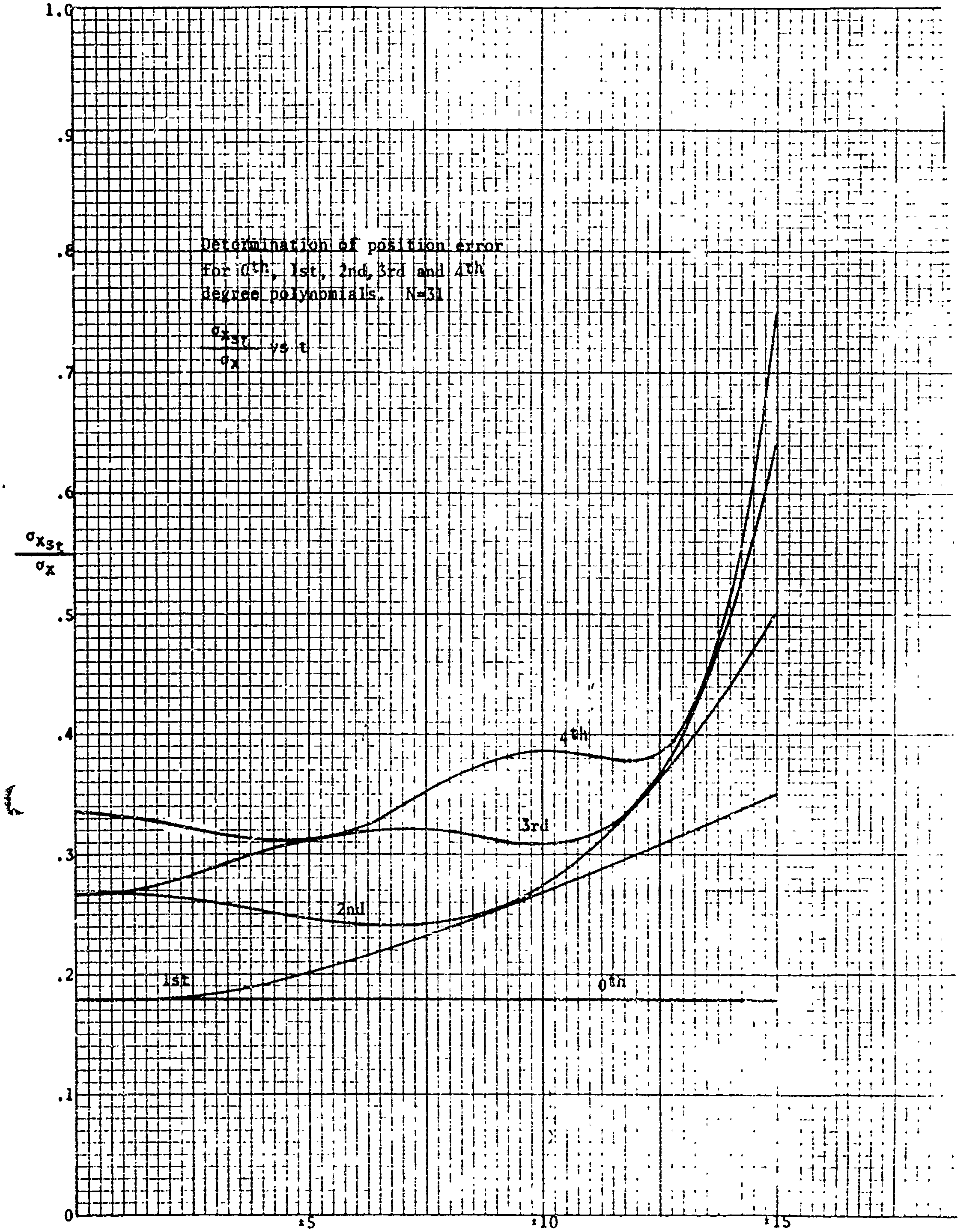
The following plots are provided as an aid in estimating variances of smoothed Positions, Velocities and Accelerations.

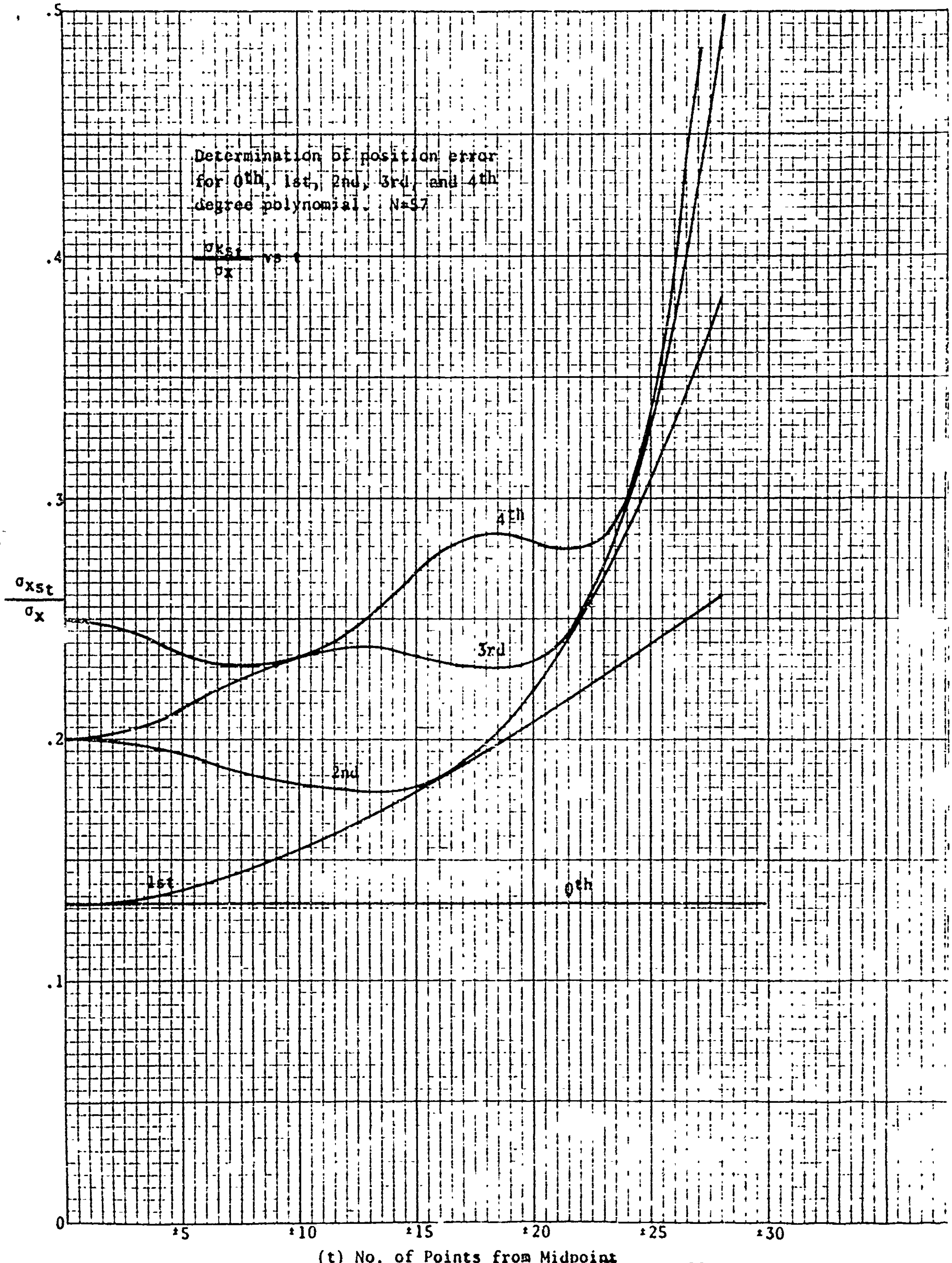


Determination of position error
for 0th, 1st, 2nd, 3rd and 4th
degree polynomials, N=9
$\frac{\sigma_{x_{S_t}}}{\sigma_x}$ vs t
4th
3rd
2nd
1st
0th
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$\frac{\sigma_{x_{S_t}}}{\sigma_x}$
±1
±2
±3
±4
(t) No. of Points from Midpoint

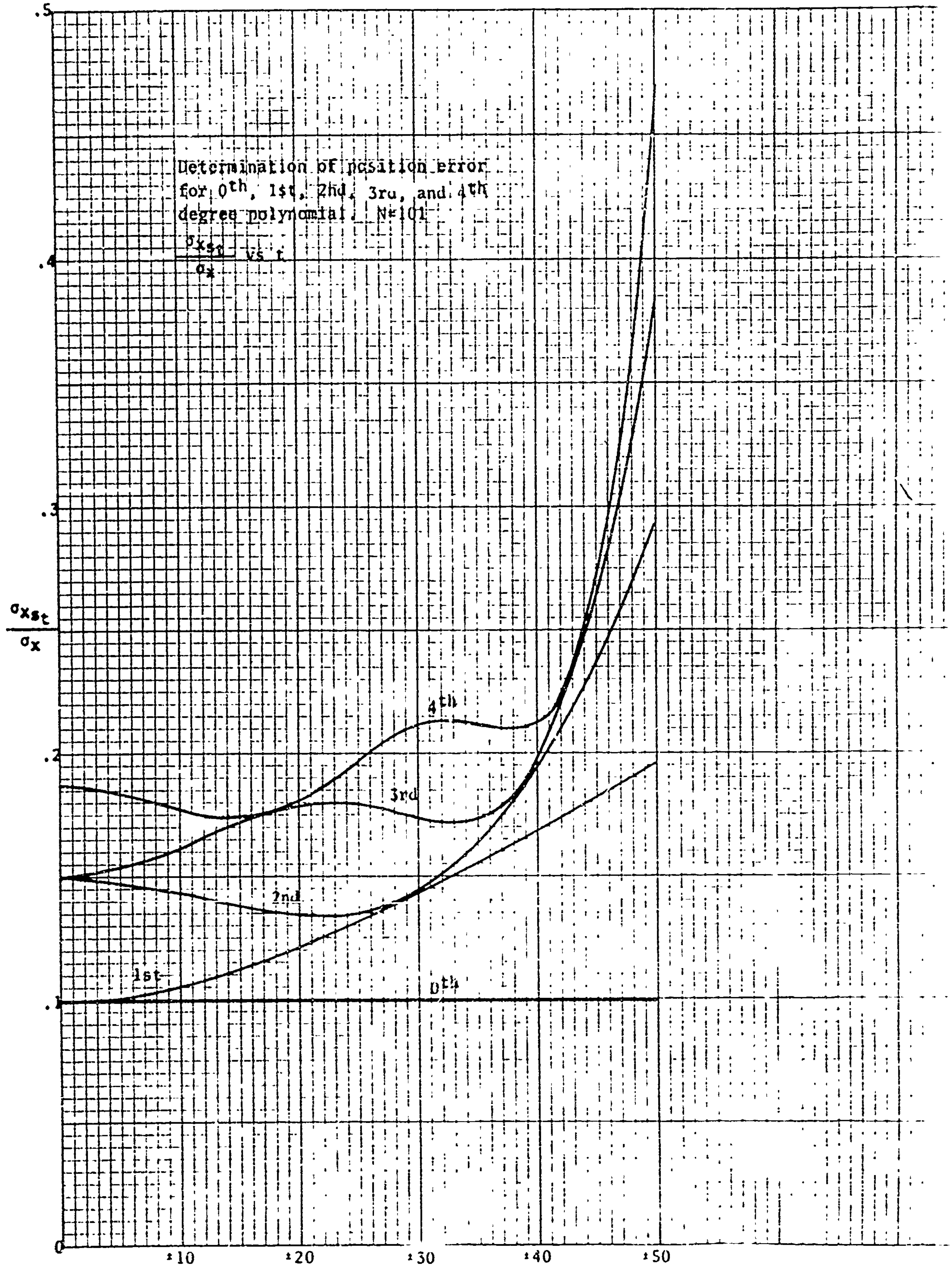


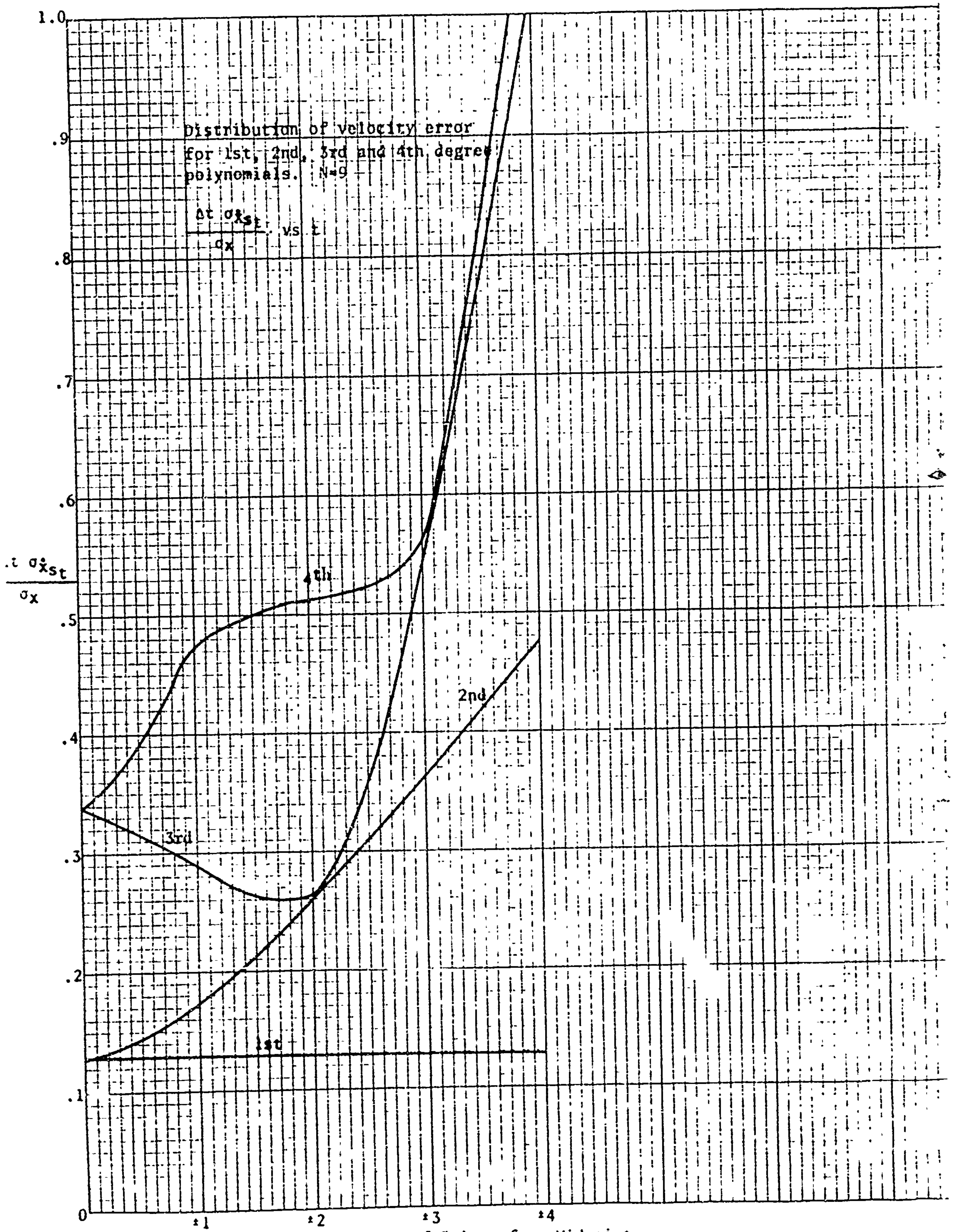
Determination of position error
for 0th, 1st, 2nd, 3rd and 4th degree
polynomials. N=17
$\frac{\sigma_{x_{st}}}{\sigma_x}$ vs t
4th
3rd
2nd
1st
0th
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
±2
±4
±6
±8
$\frac{\sigma_{x_{st}}}{\sigma_x}$
(t) No. of Points from Midpoint


Determination of position error
for 0th, 1st, 2nd, 3rd and 4th
degree polynomials. N=31
$\frac{\sigma_{x_{st}}}{\sigma_x}$ vs t
$\frac{\sigma_{x_{st}}}{\sigma_x}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
5
10
15
4th
3rd
2nd
1st
0th
(t) No. of Points from Midpoint

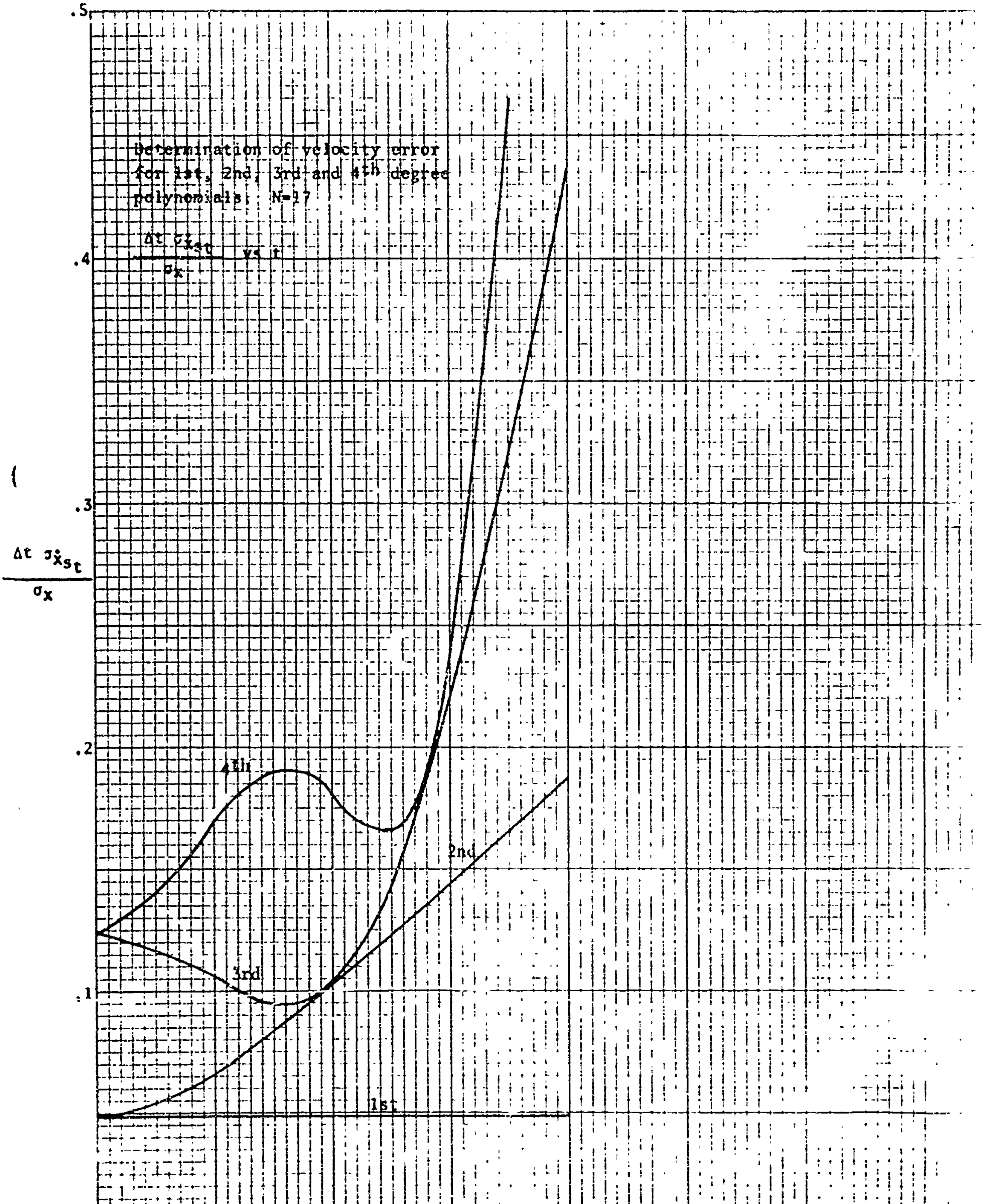


Determination of position error for 0th, 1st, 2nd, 3rd, and 4th degree polynomial. N=57
$\frac{\sigma_{x_{st}}}{\sigma_x}$ vs t
4th
3rd
2nd
1st
0th
.5
.4
.3
.2
.1
0
$\frac{\sigma_{xst}}{\sigma_x}$
±5
±10
±15
±20
±25
±30
(t) No. of Points from Midpoint

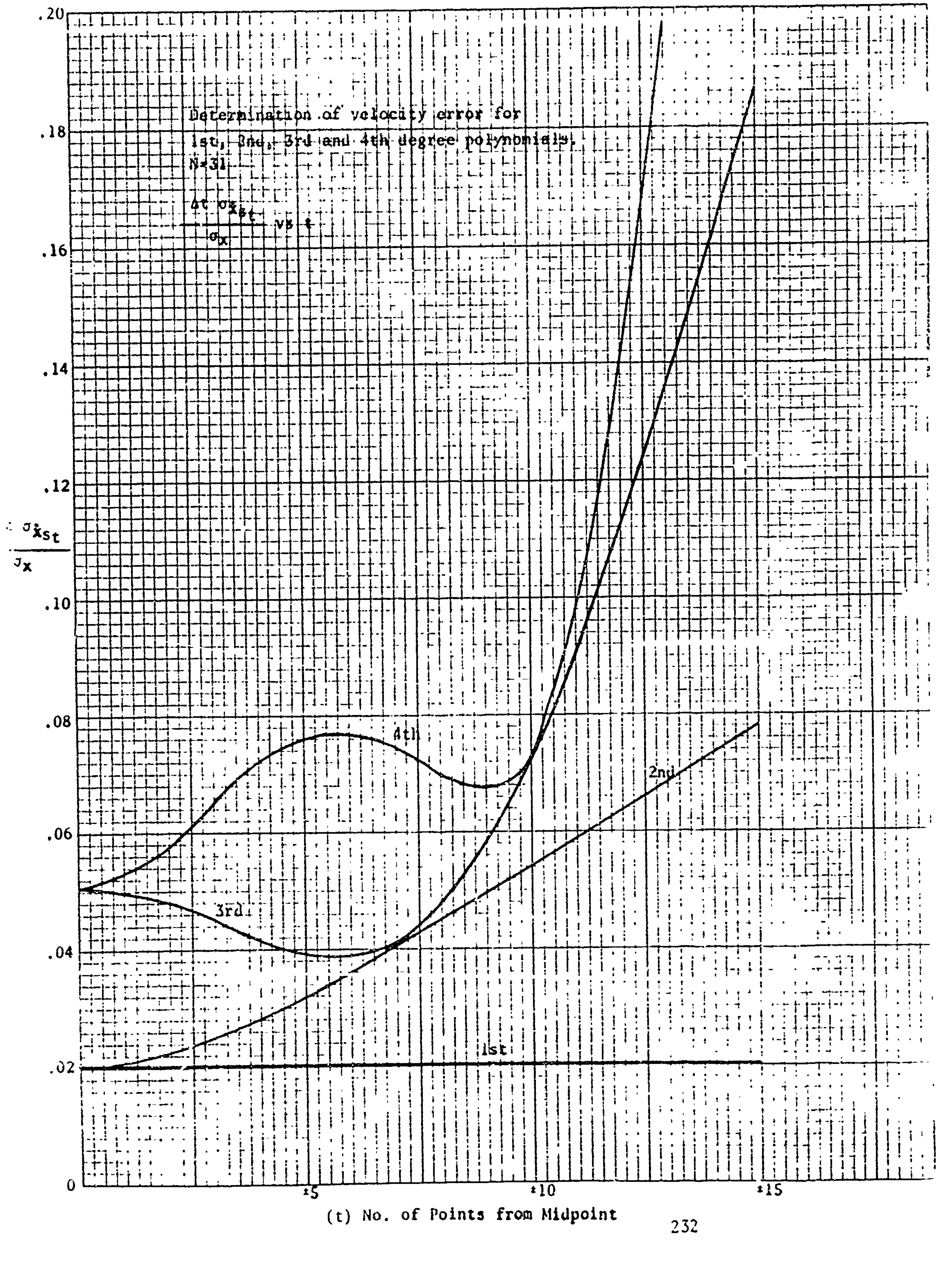

Determination of position error
for 0th, 1st, 2nd, 3rd, and 4th
degree polynomial. N=101
$\frac{\sigma_{x_{s_t}}}{\sigma_x}$ vs t
$\frac{\sigma_{x_{s_t}}}{\sigma_x}$
4th
3rd
2nd
1st
0th
.5
.4
.3
.2
.1
0
±10
±20
±30
±40
±50
(t) No. of Points from Midpoint

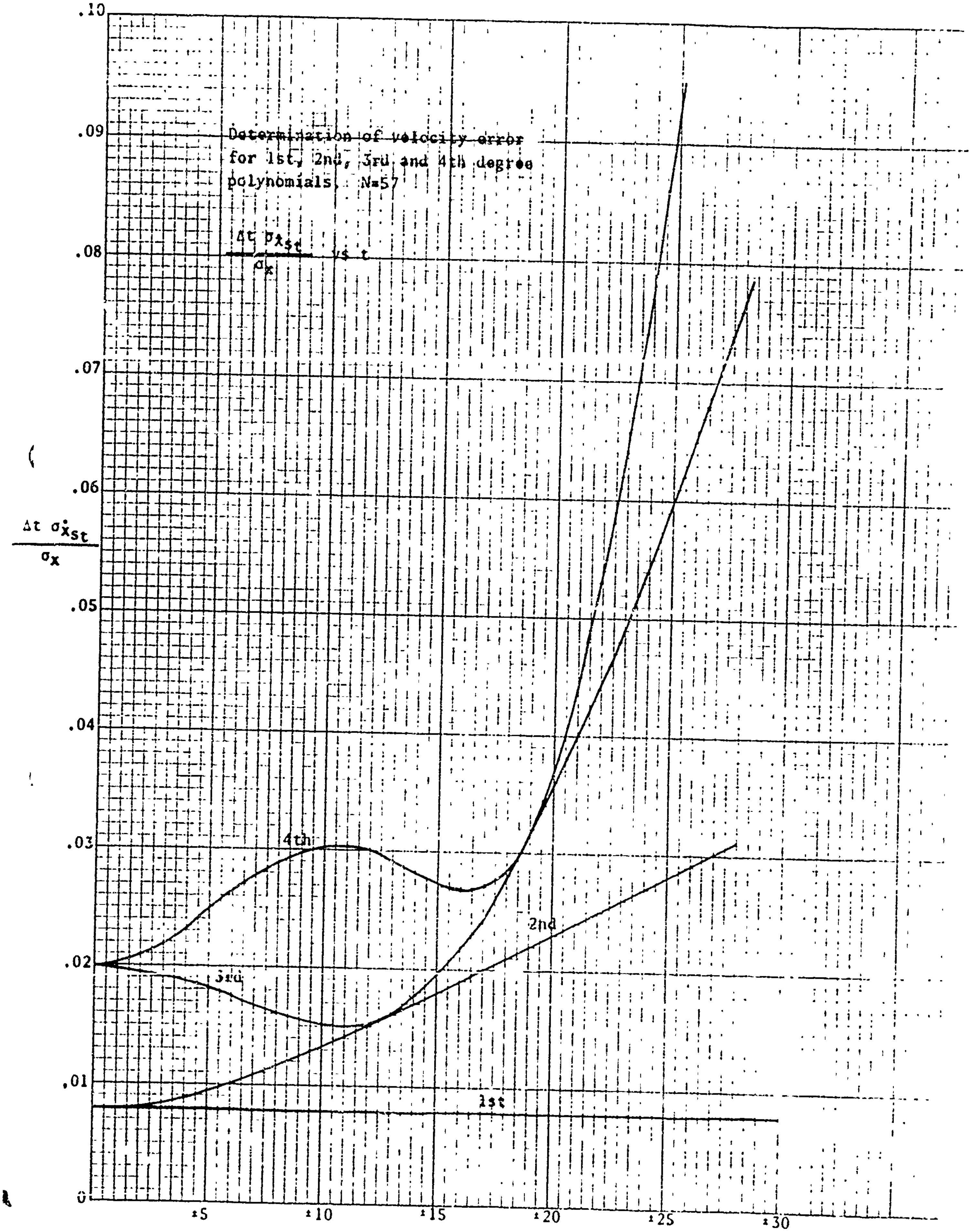


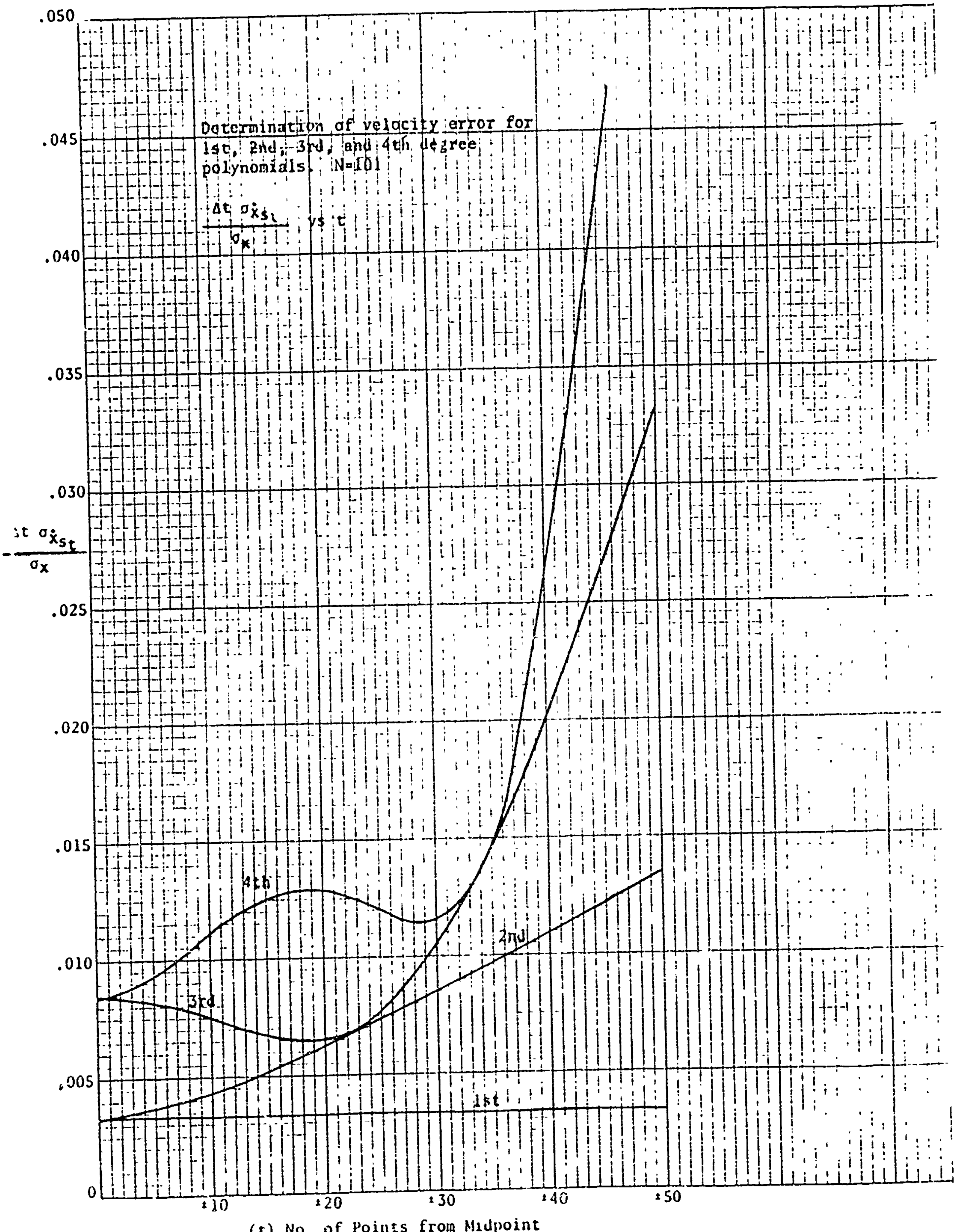
Distribution of velocity error
for 1st, 2nd, 3rd and 4th degree
polynomials. N=9
$\frac{\Delta t\ \sigma_{\dot{x}_{st}}}{\sigma_x}$ vs t
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$\frac{\Delta t\ \sigma_{\dot{x}_{st}}}{\sigma_x}$
4th
3rd
2nd
1st
±1
±2
±3
±4
(t) No. of Points from Midpoint

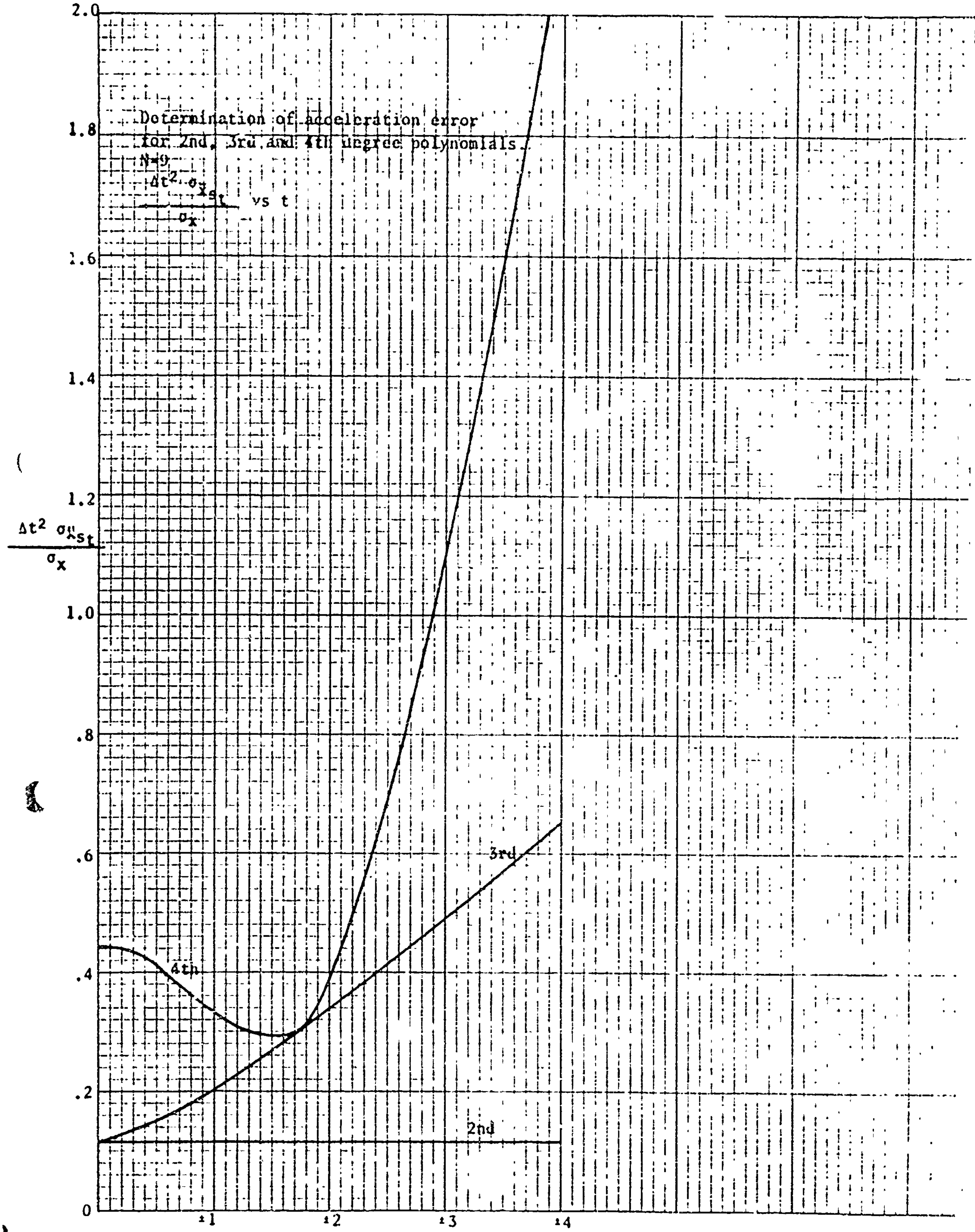


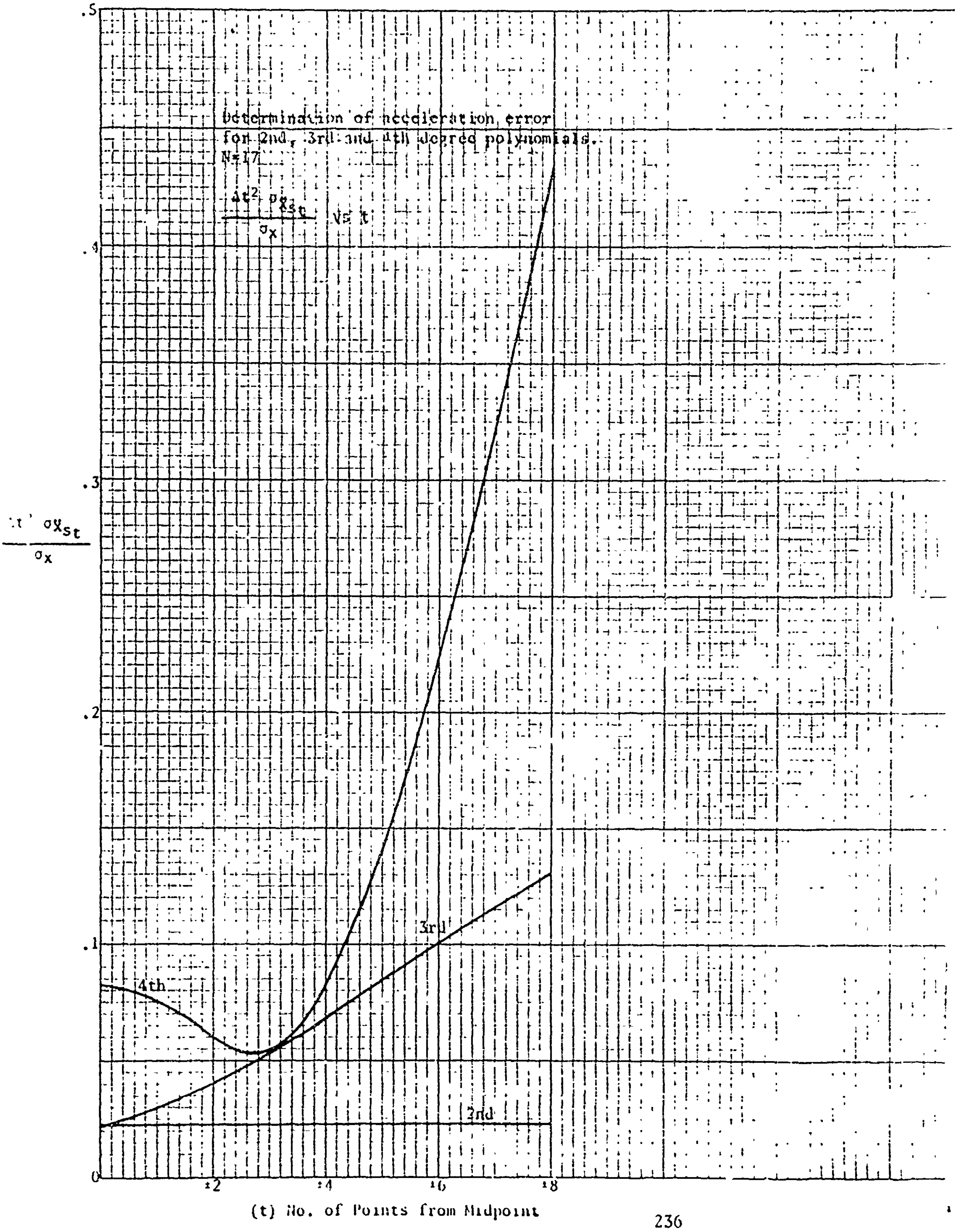
Determination of velocity error
for 1st, 2nd, 3rd and 4th degree
polynomials, N=17
$\frac{\Delta t\ \sigma_{\dot{x}_{s_t}}}{\sigma_x}$ vs t
$\frac{\Delta t\ \sigma_{\dot{x}_{s_t}}}{\sigma_x}$
.5
.4
.3
.2
.1
0
4th
3rd
2nd
1st
±2
±4
±6
±8
(t) No. of Points from Midpoint

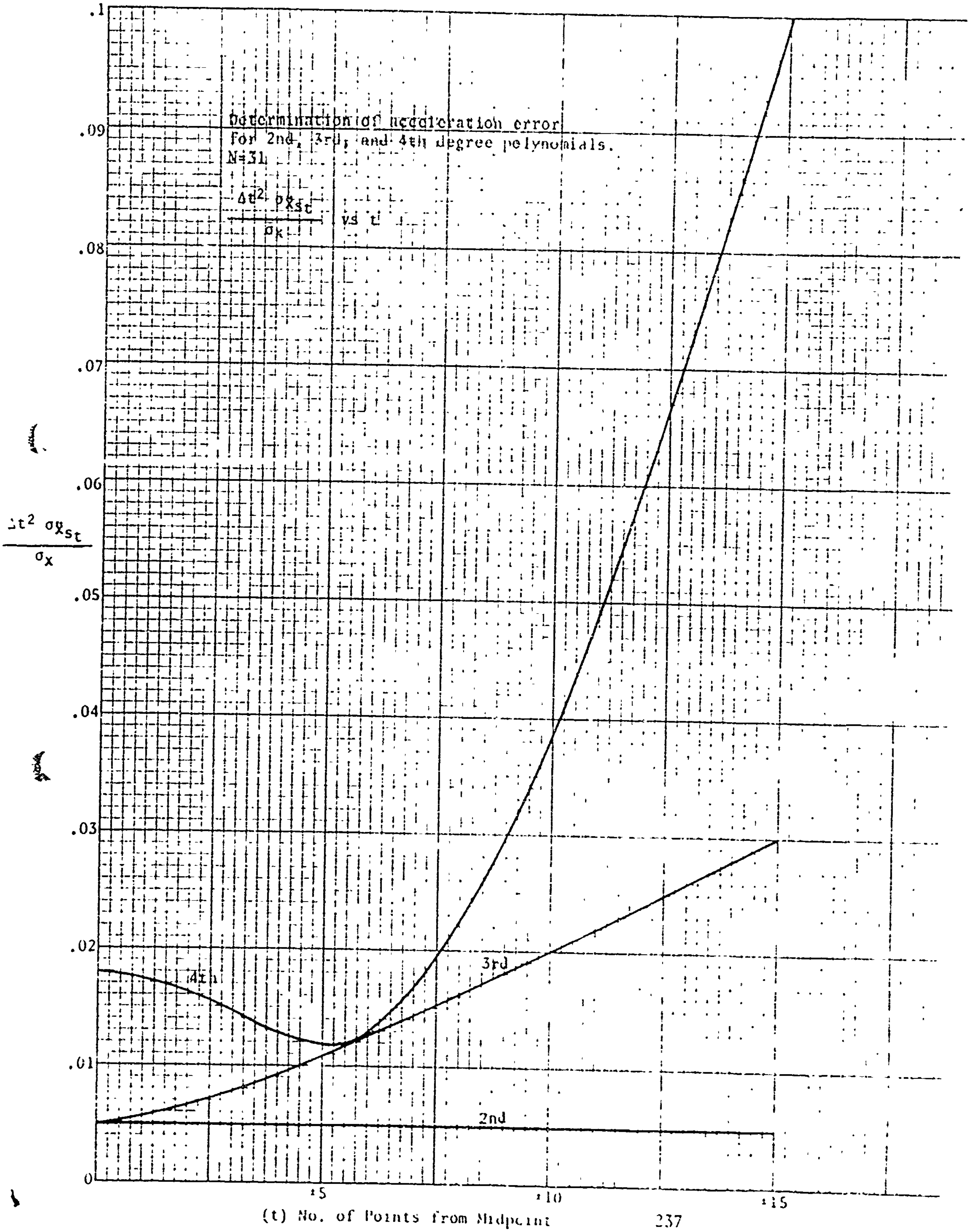

Determination of velocity error for
1st, 2nd, 3rd and 4th degree polynomials,
N=31
$\frac{\Delta t \, \sigma_{\dot{x}_{s_t}}}{\sigma_x}$ vs t
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
$\frac{\Delta t \, \sigma_{\dot{x}_{s_t}}}{\sigma_x}$
4th
3rd
2nd
1st
±5
±10
±15
(t) No. of Points from Midpoint


Determination of velocity error
for 1st, 2nd, 3rd and 4th degree
polynomials. N=57
$\frac{\Delta t\ \sigma_{\dot{x}_{st}}}{\sigma_x}$ vs t
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
$\frac{\Delta t\ \sigma_{\dot{x}_{st}}}{\sigma_x}$
4th
3rd
2nd
1st
±5
±10
±15
±20
±25
±30
(t) No. of Points from Midpoint

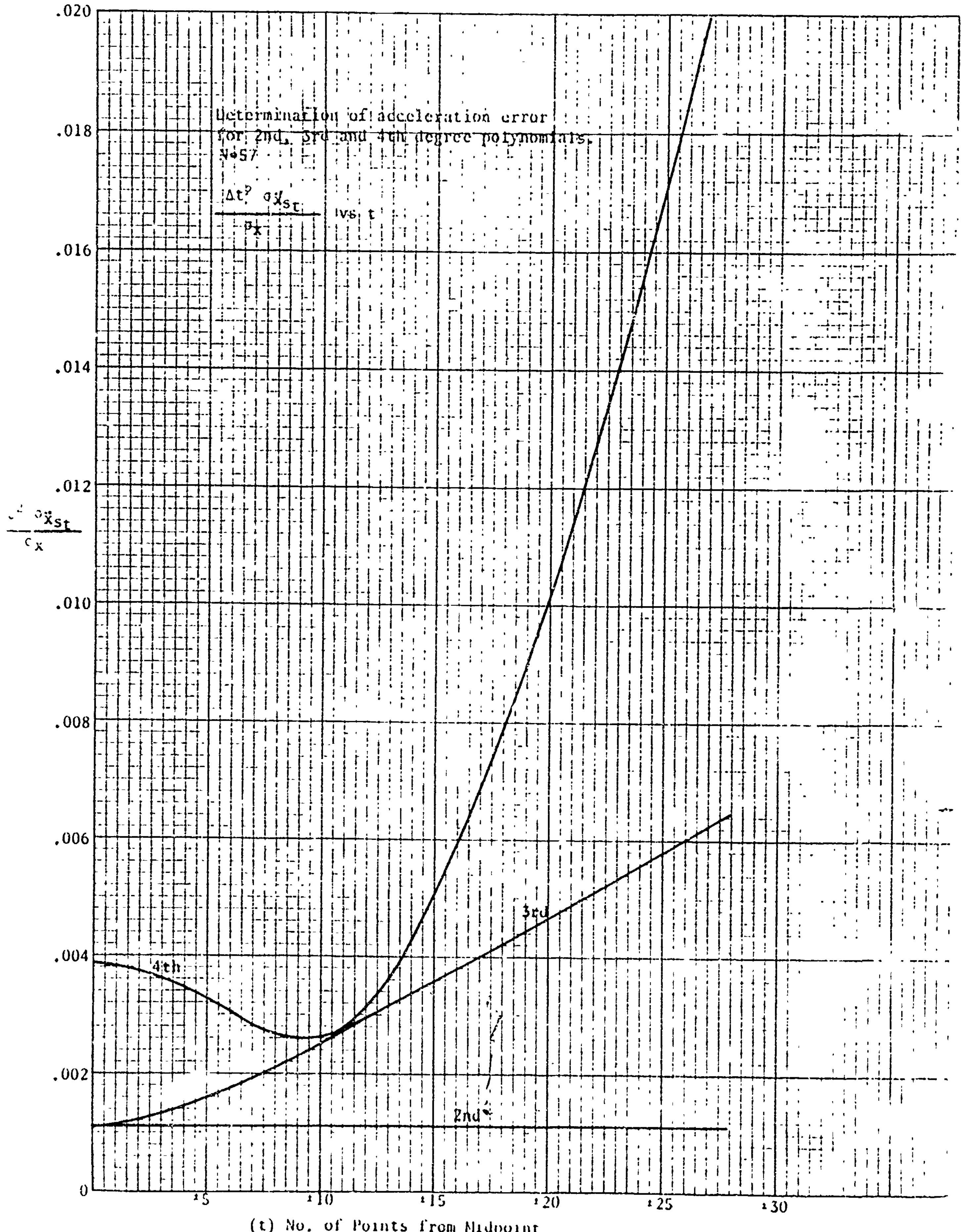

Determination of velocity error for
1st, 2nd, 3rd, and 4th degree
polynomials. N=101
$\frac{\Delta t\, \sigma_{\dot{x}_{s_t}}}{\sigma_x}$ vs t
.050
.045
.040
.035
.030
.025
.020
.015
.010
.005
0
$\frac{\Delta t\, \sigma_{\dot{x}_{s_t}}}{\sigma_x}$
4th
3rd
2nd
1st
±10
±20
±30
±40
±50
(t) No. of Points from Midpoint


Determination of acceleration error
for 2nd, 3rd and 4th degree polynomials
N=9
$\frac{\Delta t^2 \sigma_{\ddot{x}_{s_t}}}{\sigma_x}$ vs t
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
$\frac{\Delta t^2 \sigma_{\ddot{x}_{s_t}}}{\sigma_x}$
4th
3rd
2nd
±1
±2
±3
±4
(t) No. of points from Midpoint

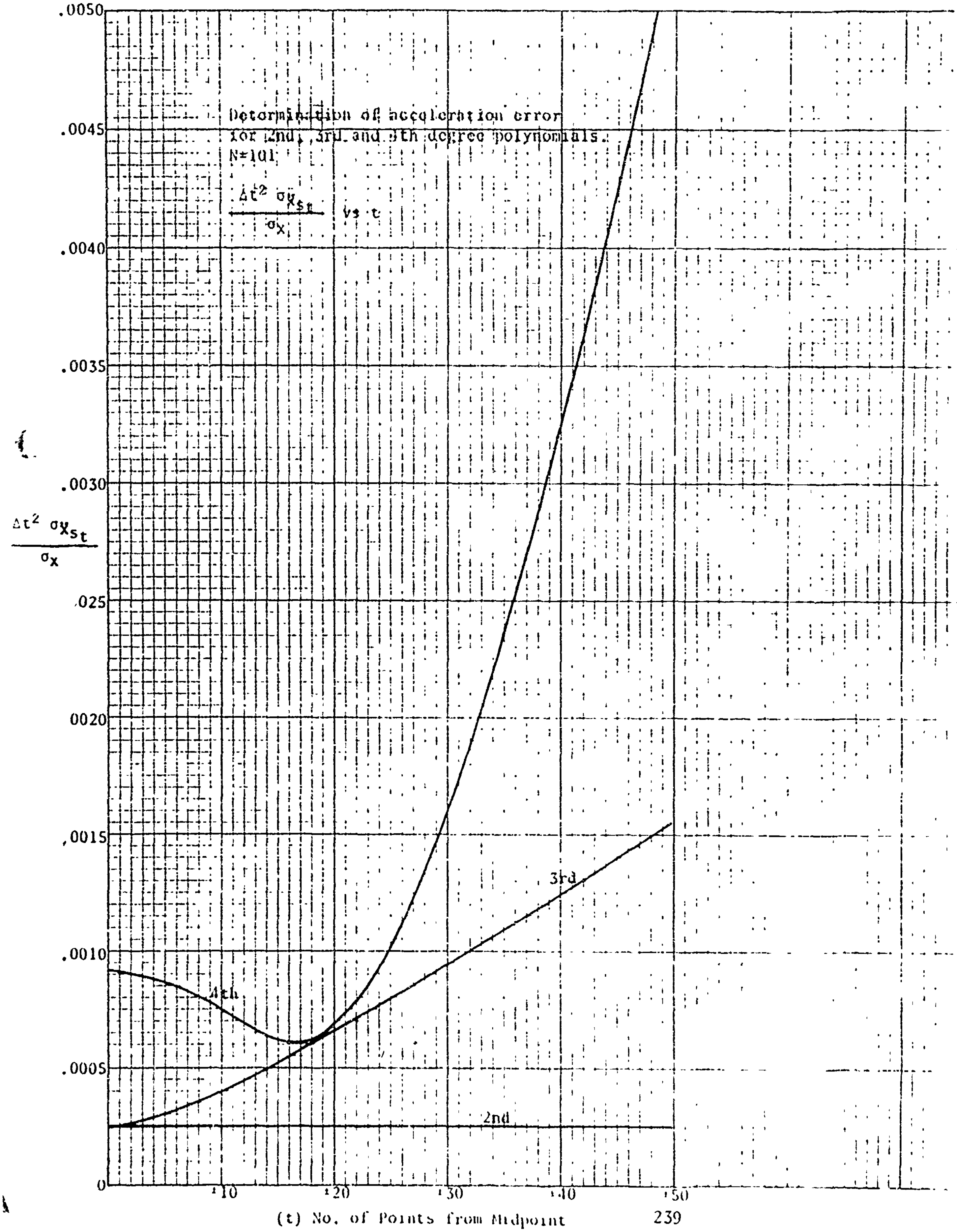

Determination of acceleration error
for 2nd, 3rd and 4th degree polynomials.
N=17
$\frac{\Delta t^2\ \sigma \ddot{x}_{st}}{\sigma_x}$ vs t
4th
3rd
2nd
.5
.4
.3
.2
.1
0
±2
±4
±6
±8
(t) No. of Points from Midpoint



Determination of acceleration error
for 2nd, 3rd, and 4th degree polynomials.
N=31
$\frac{\Delta t^2\ \sigma \ddot{x}_{s_t}}{\sigma_x}$ vs t
$\frac{\Delta t^2\ \sigma \ddot{x}_{s_t}}{\sigma_x}$
.1
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
4th
3rd
2nd
±5
±10
±15
(t) No. of Points from Midpoint



Determination of acceleration error
for 2nd, 3rd and 4th degree polynomials,
N=57
$\frac{\Delta t^2 \, \sigma \ddot{x}_{s_t}}{\sigma_x}$ vs. t
.020
.018
.016
.014
.012
.010
.008
.006
.004
.002
0
$\frac{\Delta t^2 \, \sigma \ddot{x}_{s_t}}{\sigma_x}$
4th
3rd
2nd
±5
±10
±15
±20
±25
±30
(t) No. of Points from Midpoint



Determination of acceleration error
for 2nd, 3rd and 4th degree polynomials.
N=101
$\frac{\Delta t^2 \sigma x_{s_t}}{\sigma_x}$ vs t
.0050
.0045
.0040
.0035
.0030
.0025
.0020
.0015
.0010
.0005
0
$\frac{\Delta t^2 \sigma x_{s_t}}{\sigma_x}$
4th
3rd
2nd
10
20
30
40
50
(t) No. of Points from Midpoint

ORTHOGONAL POLYNOMIAL REFERENCES

Kendall, M.G. and Stuart, H. The Advanced Theory of Statistics, Vol. II, New York, Hafner Publishing Co. 1961

D. VELOCITY AND ACCELERATION

III Functions of Velocity and Acceleration

FUNCTIONS OF VELOCITY AND ACCELERATION

The moving arc smoothing program and the orthogonal polynomial program compute smoothed positions, velocity components and acceleration components. The standard deviations of these data are also obtained. From these data other derivative data may be computed. These data include trajectory angles, the rate of change of the trajectory angles, tangential and normal acceleration components, total velocity, radius of curvature and rate of turn.

This report includes the derivation of the equations for these data and for their standard deviations.

Velocity:

The coordinates (x, y, z) of a point, P, on a curve are expressed as functions of a third variable, or parameter, t, in the form

$$x = f_0(t)$$
$$y = f_1(t) \qquad (1)$$
$$z = f_2(t).$$

When the parameter, t, is time, the functions, $f_i(t)$ are continuous and if t varies continuously then the point (x, y, z) will trace the curve or path. We then have a curvilinear motion and equations (1) are called the equations of motion.

The velocity or time rate of change of the distance of the point, P, at any instant is determined by its velocity components.

$$\frac{dx}{dt} = \dot{x}$$
$$\frac{dy}{dt} = \dot{y} \qquad (2)$$
$$\frac{dz}{dt} = \dot{z}.$$

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The magnitude of the velocity, as seen in figure 1, is given by

$$V_t^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{3}$$

and

$$V_t = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} \tag{4}$$

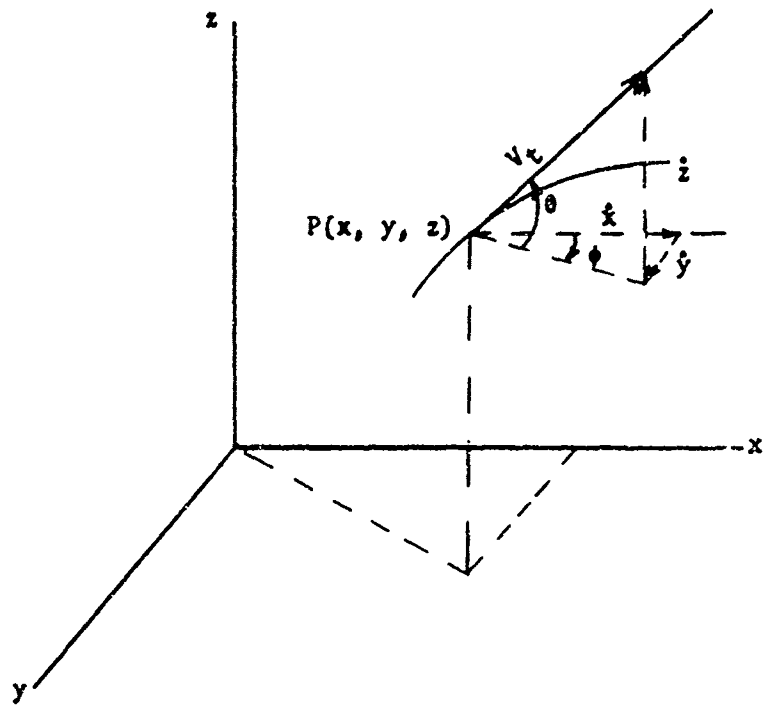

FIGURE 1.

In figure (1)

$\dot{x}, \dot{y}, \dot{z}$ = velocity component vectors

V_t = tangential velocity vector

P(x, y, z) = point on the trajectory

θ = trajectory elevation angle

ϕ = trajectory azimuth angle

Trajectory angles:

The trajectory angles, ϕ and θ, are defined as the azimuth and elevation angles, respectively, of the tangential velocity vector for a point, P, on a trajectory. Figure (1) illustrates the situation at the point, P, on the trajectory in the x, y, z coordinate system.

It can easily be seen that

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) \tag{5}$$

$$\theta = \tan^{-1}\left[\frac{\dot{z}}{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}}\right]. \tag{6}$$

Total acceleration:

The rate of change of the velocity with respect to time is called acceleration. The total acceleration, A_s, may be resolved into components parallel to the coordinate axes in the same manner as velocity components were determined.

That is,

$$\frac{d\dot{x}}{dt} = \ddot{x}$$

$$\frac{d\dot{y}}{dt} = \ddot{y} \tag{7}$$

$$\frac{d\dot{z}}{dt} = \ddot{z}$$

and

$$A_S = (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)^{\frac{1}{2}} \tag{8}$$

This acceleration vector is not, like the velocity vector, always directed along the tangent to the path. It may also be resolved into two components, tangential, A_t, and normal, A_N, giving the total acceleration

$$A_S = (A_N^2 + A_t^2)^{\frac{1}{2}} \tag{9}$$

Tangential acceleration:

The tangential component is in the direction of the tangent to the curve and is equal to the time rate of change of speed at the point, P.

$$A_t = \frac{d\,V_t}{dt}$$

$$A_t = \frac{d(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{½}}{dt} = \frac{\dot{x}\frac{d\dot{x}}{dt} + \dot{y}\frac{d\dot{y}}{dt} + \dot{z}\frac{d\dot{z}}{dt}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{½}} \tag{10}$$

and using equation (7)

$$A_t = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{½}} . \tag{11}$$

Normal acceleration:

This component is normal to the tangent at the point, P, and directed toward the center of curvature (Fig. 2).

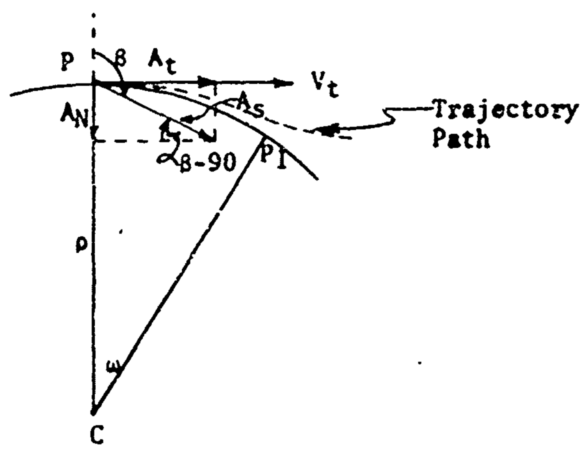

FIGURE 2.

PP_1 is an arc of the circle of curvature through point P.

From equation (9) and Fig. 2 we see that

$$A_N = (A_s{}^2 - A_t{}^2)^{½} . \tag{12}$$

Substituting equations (8) and (11) into equation (12) yields

$$A_N = \left[\frac{(\dot{y}\ \ddot{z} - \dot{z}\ \ddot{y})^2 + (\dot{z}\ \ddot{x} - \dot{x}\ \ddot{z})^2 + (\dot{x}\ \ddot{y} - \dot{y}\ \ddot{x})^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right]^{\frac{1}{2}}. \tag{13}$$

Radius of curvature:

The shape of a curve at a point depends upon the rate of change of direction. This rate is called the curvature at the point. The radius of curvature at the point is defined as the reciprocal of the curvature and may be found in the following manner:

Referring to Figure 2, ρ is the radius of curvature at point P and

$$\sin\ (\beta - 90°) = \frac{A_N}{A_S}. \tag{14}$$

Since

$$\sin\ (\beta - 90°) = -\cos\ \beta$$

then

$$\cos\ \beta = -\frac{A_N}{A_s}. \tag{15}$$

The cosine of the angle, β, may also be found from the following equation:

$$\cos\ \beta = \lambda_1\ \lambda_2 + \mu_1\ \mu_2 + \nu_1\ \nu_2. \tag{16}$$

where λ_1, μ_1, ν_1 are the direction cosines of the line CP,

$$\lambda_1 = \frac{x}{\rho}$$

$$\mu_1 = \frac{y}{\rho} \tag{17}$$

$$\nu_1 = \frac{z}{\rho}$$

and λ_2, μ_2, ν_2 are the direction cosines of the vector, A_S,

$$\lambda_2 = \frac{\ddot{x}}{A_s}$$

$$\mu_2 = \frac{\ddot{y}}{A_s} \tag{18}$$

$$\nu_2 = \frac{\ddot{z}}{A_s}.$$

Then

$$\cos\beta = \frac{x\ddot{x} + y\ddot{y} + z\ddot{z}}{\rho A_s} \tag{19}$$

and from equation (15)

$$-\frac{A_N}{A_s} = \frac{x\ddot{x} + y\ddot{y} + z\ddot{z}}{\rho A_s}$$

$$-\rho = \frac{x\ddot{x} + y\ddot{y} + z\ddot{z}}{A_N}. \tag{20}$$

Since

$$\rho^2 = x^2 + y^2 + z^2 \tag{21}$$

and ρ is constant from P to P_1

then $$\frac{d\rho}{dt} = 0 = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\rho}$$

and $$\frac{d^2\rho}{dt^2} = 0 = \frac{x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2}{\rho} \tag{22}$$

or $$x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0$$

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{23}$$

Substituting equation (3) into equation (23) yields:

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -V_t^2. \tag{24}$$

Then using this in equation (20) gives:

$$\rho = \frac{V_t^2}{A_N}. \tag{25}$$

Using equations (3) and (13) the radius of curvature may also be found from

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}}{[(\dot{y}\,\ddot{z} - \dot{z}\,\ddot{y})^2 + (\dot{z}\,\ddot{x} - \dot{x}\,\ddot{z})^2 + (\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})^2]^{1/2}}. \tag{26}$$

Rate of turn:

In Figure (2) let the arc length PP_1 be denoted by s, then

$$s = \omega\rho. \tag{27}$$

The time rate of change of the arc length is

$$\frac{ds}{dt} = \rho\left(\frac{d\omega}{dt}\right) + \omega\left(\frac{d\rho}{dt}\right) \tag{28}$$

and since

$$\frac{d\rho}{dt} = 0$$

$$\frac{ds}{dt} = \rho\left(\frac{d\omega}{dt}\right)$$

then

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{\dot{s}}{\rho}. \tag{29}$$

The arc length, s, may also be found from the following equation

$$s = \int_{t_1}^{t}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt \tag{30}$$

and the derivative of s with respect to t is

$$\dot{s} = \frac{ds}{dt} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} \tag{31}$$

$$\dot{s} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} = V_t. \tag{32}$$

Substituting equation (32) in equation (29) gives the angular rate at which the arc is changing or the rate of turn

$$\dot{\omega} = \frac{V_t}{\rho}, \tag{33}$$

Rate of change of trajectory angles:

The time rate of change of the azimuth trajectory angle is found by taking the derivative of equation (5) with respect to time.

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right), \tag{5}$$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^2 + \dot{y}^2}. \tag{34}$$

The time rate of change of the elevation trajectory angle is found in the same manner.

$$\theta = \tan^{-1}\left[\frac{\dot{z}}{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}}\right]. \tag{6}$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\ \ddot{z} - \dot{z}(\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y})(\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}}}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \tag{35}$$

$$\dot{\theta} = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\ \ddot{z} - \dot{z}(\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y})(\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}}}{V_t^2}. \tag{36}$$

$\dot{\phi}$ and $\dot{\theta}$ are in radians per second.

Standard deviations:

In the following derivations the variances of the velocity and acceleration components are those variances obtained from the smoothed position reduction.

Trajectory angles:

The azimuth angle is

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right). \tag{5}$$

The variance of the angle is found in the following manner.

Since

$$\phi = f(\dot{x}, \dot{y}) \tag{37}$$

then the true azimuth angle, $\phi + \Delta\phi$, may be expressed in a Taylor's series expansion

$$(\phi + \Delta\phi) = f(\dot{x} + \Delta\dot{x}, \dot{y} + \Delta\dot{y})$$

$$= f(\dot{x}, \dot{y}) + f'(\dot{x}, \dot{y}) \frac{(\Delta\dot{x} + \Delta\dot{y})}{1!} + f''(\dot{x}, \dot{y}) \frac{(\Delta\dot{x} + \Delta\dot{y})^2}{2!} + \cdots \tag{38}$$

where $\Delta\dot{x}$, $\Delta\dot{y}$ are residuals and the primes indicate differentiation with respect to $\dot{x}$ and $\dot{y}$.

Neglecting higher order terms in equation (38) gives

$$\Delta\phi = \frac{\partial\phi}{\partial\dot{x}} \Delta\dot{x} + \frac{\partial\phi}{\partial\dot{y}} \Delta\dot{y} \tag{39}$$

then

$$\Sigma(\Delta\phi)^2 = \left(\frac{\partial\phi}{\partial\dot{x}}\right)^2 \Sigma(\Delta\dot{x})^2 + \left(\frac{\partial\phi}{\partial\dot{y}}\right)^2 \Sigma(\Delta\dot{y})^2 + 2\left(\frac{\partial\phi}{\partial\dot{x}}\right)\left(\frac{\partial\phi}{\partial\dot{y}}\right)\Sigma\Delta\dot{x}\,\Delta\dot{y}$$

If $\Delta\dot{x}$ and $\Delta\dot{y}$ are random then $\Sigma\Delta\dot{x}\,\Delta\dot{y}$ approaches zero and therefore may be neglected,

$$\Sigma(\Delta\phi)^2 = \left(\frac{\partial\phi}{\partial\dot{x}}\right)^2 \Sigma(\Delta\dot{x})^2 + \left(\frac{\partial\phi}{\partial\dot{y}}\right)^2 \Sigma(\Delta\dot{y})^2. \tag{40}$$

Variance is defined as the sum of the squares of the residuals divided by the number of observations; therefore we know

$$\sigma_\phi^2 = \frac{\Sigma(\Delta\phi)^2}{N} \text{ or } \Sigma(\Delta\phi)^2 = N\sigma_\phi^2 \tag{41}$$

and

$$\sigma_{\dot{x}}^2 = \frac{\Sigma(\Delta\dot{x})^2}{N} \text{ or } \Sigma(\Delta\dot{x})^2 = N\sigma_{\dot{x}}^2 \tag{42}$$

$$\sigma_{\dot{y}}^2 = \frac{\Sigma(\Delta\dot{y})^2}{N} \text{ or } \Sigma(\Delta\dot{y})^2 = N\sigma_{\dot{y}}^2. \tag{43}$$

Substituting these equations in equation (40) yields

$$\sigma_\phi^2 = \left(\frac{\partial\phi}{\partial\dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial\phi}{\partial\dot{y}}\right)^2 \sigma_{\dot{y}}^2 \tag{44}$$

where

$$\left(\frac{\partial\phi}{\partial\dot{x}}\right)^2 = \frac{\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^2} \tag{45}$$

$$\left(\frac{\partial\phi}{\partial\dot{y}}\right)^2 = \frac{\dot{x}^2}{(\dot{x}^2 + \dot{y}^2)^2} \tag{46}$$

therefore

$$\sigma_\phi^2 = \frac{\dot{y}^2\,\sigma_{\dot{x}}^2 + \dot{x}^2\,\sigma_{\dot{y}}^2}{(\dot{x}^2 + \dot{y}^2)^2} \tag{47}$$

and the standard deviation of the azimuth angle is

$$\sigma_\phi = \left(\frac{\dot{y}^2\,\sigma_{\dot{x}}^2 + \dot{x}^2\,\sigma_{\dot{y}}^2}{(\dot{x}^2 + \dot{y}^2)^2}\right)^{1/2}. \tag{48}$$

The standard deviation of the elevation trajectory angle is found as follows:

$$\theta = \tan^{-1}\left[\frac{\dot{z}}{(\dot{x}^2 + \dot{y}^2)^{1/2}}\right]. \tag{6}$$

The variance is derived in the same manner as the variance for the azimuth angle and

$$\sigma_\theta^2 = \left(\frac{\partial\theta}{\partial\dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial\theta}{\partial\dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial\theta}{\partial\dot{z}}\right)^2 \sigma_{\dot{z}}^2 \tag{49}$$

where

$$\left(\frac{\partial\theta}{\partial\dot{x}}\right)^2 = \frac{\dot{x}^2\ \dot{z}^2}{(\dot{x}^2 + \dot{y}^2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^2} \tag{50}$$

$$\left(\frac{\partial\theta}{\partial\dot{y}}\right)^2 = \frac{\dot{y}^2\ \dot{z}^2}{(\dot{x}^2 + \dot{y}^2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^2} \tag{51}$$

$$\left(\frac{\partial\theta}{\partial\dot{z}}\right)^2 = \frac{\dot{x}^2 + \dot{y}^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^2} \tag{52}$$

then

$$\sigma_\theta^2 = \frac{\left(\frac{\dot{x}^2\ \dot{z}^2}{\dot{x}^2 + \dot{y}^2}\right)\sigma_{\dot{x}}^2 + \left(\frac{\dot{y}^2\ \dot{z}^2}{\dot{x}^2 + \dot{y}^2}\right)\sigma_{\dot{y}}^2 + (\dot{x}^2 + \dot{y}^2)\ \sigma_{\dot{z}}^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^2}. \tag{53}$$

At this point we will make the following substitutions which will simplify the writing of this equation:

$$\dot{G} = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \tag{54}$$

$$\sigma_{\dot{G}}^2 = \frac{\dot{x}^2\ \sigma_{\dot{x}}^2 + \dot{y}^2\ \sigma_{\dot{y}}^2}{(\dot{x}^2 + \dot{y}^2)}. \tag{55}$$

Using these substitutions, and equation (3), equation (53) becomes

$$\sigma_\theta^2 = \frac{\dot{z}^2\ \sigma_{\dot{G}}^2 + \dot{G}^2\ \sigma_{\dot{z}}^2}{V_t^4}. \tag{56}$$

The standard deviation of the elevation trajectory angle is

$$\sigma_\theta = \left(\frac{\dot{z}^2\ \sigma_{\dot{G}}^2 + \dot{G}^2\sigma_{\dot{z}}^2}{V_t^4}\right)^{\frac{1}{2}}. \tag{57}$$

Velocity:

$$V_t = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} \tag{4}$$

The variance of the velocity is derived as before and is given by the following equation:

$$\sigma^2_{V_t} = \left(\frac{\partial V_t}{\partial \dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial V_t}{\partial \dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial V_t}{\partial \dot{z}}\right)^2 \sigma_{\dot{z}}^2 \tag{58}$$

where

$$\left(\frac{\partial V_t}{\partial \dot{x}}\right)^2 = \frac{\dot{x}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{\dot{x}^2}{V_t^2} \tag{59}$$

$$\left(\frac{\partial V_t}{\partial \dot{y}}\right)^2 = \frac{\dot{y}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad \frac{\dot{y}^2}{V_t^2} \tag{60}$$

$$\left(\frac{\partial V_t}{\partial \dot{z}}\right)^2 = \frac{\dot{z}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{\dot{z}^2}{V_t^2}. \tag{61}$$

The standard deviation of the velocity is

$$\sigma_{V_t} = \left(\frac{\dot{x}^2 \sigma_{\dot{x}}^2 + \dot{y}^2 \sigma_{\dot{y}}^2 + \dot{z}^2 \sigma_{\dot{z}}^2}{V_t^2}\right)^{\frac{1}{2}}. \tag{62}$$

Tangential acceleration:

$$A_t = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}}. \tag{11}$$

$$= \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}}{V_t}.$$

The variance of the tangential acceleration becomes

$$\sigma^2_{A_t} = \left(\frac{\partial A_t}{\partial \dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial A_t}{\partial \dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial A_t}{\partial \dot{z}}\right)^2 \sigma_{\dot{z}}^2 + \left(\frac{\partial A_t}{\partial \ddot{x}}\right)^2 \sigma_{\ddot{x}}^2$$

$$+ \left(\frac{\partial A_t}{\partial \ddot{y}}\right)^2 \sigma_{\ddot{y}}^2 + \left(\frac{\partial A_t}{\partial \ddot{z}}\right)^2 \sigma_{\ddot{z}}^2 \tag{63}$$

where

$$\left(\frac{\partial A_t}{\partial \dot{x}}\right)^2 = \left(\frac{V_t \ddot{x} - \dot{x} A_t}{V_t^2}\right)^2 = (P_x)^2 \tag{64}$$

$$\left(\frac{\partial A_t}{\partial \dot{y}}\right)^2 = \left(\frac{V_t \ddot{y} - \dot{y} A_t}{V_t^2}\right)^2 = (P_y)^2 \tag{65}$$

$$\left(\frac{\partial A_t}{\partial \dot{z}}\right)^2 = \left(\frac{V_t \ddot{z} - \dot{z} A_t}{V_t^2}\right)^2 = (P_z)^2 \tag{66}$$

$$\left(\frac{\partial A_t}{\partial \ddot{x}}\right)^2 = \frac{\dot{x}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{\dot{x}^2}{V_t^2} \tag{67}$$

$$\left(\frac{\partial A_t}{\partial \ddot{y}}\right)^2 = \frac{\dot{y}^2}{V_t^2} \tag{68}$$

$$\left(\frac{\partial A_t}{\partial \ddot{z}}\right)^2 = \frac{\dot{z}^2}{V_t^2}. \tag{69}$$

Substituting in equation (63) yields

$$\sigma_{A_t}^2 = (P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2 + \frac{\dot{x}^2 \sigma_{\ddot{x}}^2 + \dot{y}^2 \sigma_{\ddot{y}}^2 + \dot{z}^2 \sigma_{\ddot{z}}^2}{V_t^2}. \tag{70}$$

The standard deviation of the tangential acceleration is

$$\sigma_{A_t} = \left[(P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2 + \frac{\dot{x}^2 \sigma_{\ddot{x}}^2 + \dot{y}^2 \sigma_{\ddot{y}}^2 + \dot{z}^2 \sigma_{\ddot{z}}^2}{V_t^2}\right]^{\frac{1}{2}}. \tag{71}$$

Normal acceleration:

$$A_N = (A_s^2 - A_t^2)^{\frac{1}{2}}. \tag{12}$$

The variance of this acceleration is given by

$$\sigma_{A_N}^2 = \left(\frac{\partial A_N}{\partial \dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial A_N}{\partial \dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial A_N}{\partial \dot{z}}\right)^2 \sigma_{\dot{z}}^2 + \left(\frac{\partial A_N}{\partial \ddot{x}}\right)^2 \sigma_{\ddot{x}}^2$$

$$+ \left(\frac{\partial A_N}{\partial \ddot{y}}\right)^2 \sigma_{\ddot{y}}^2 + \left(\frac{\partial A_N}{\partial \ddot{z}}\right)^2 \sigma_{\ddot{z}}^2 \tag{72}$$

where

$$\left(\frac{\partial A_N}{\partial \dot{x}}\right)^2 = \left(\frac{-A_t \quad \frac{\partial A_t}{\partial \dot{x}}}{(A_s^2 - A_t^2)^{1/2}}\right)^2 = \left(\frac{-A_t}{A_N}\right)^2 \left(\frac{\partial A_t}{\partial \dot{x}}\right)^2 \tag{73}$$

and using equation (64)

$$\left(\frac{\partial A_N}{\partial \dot{x}}\right)^2 = \left(\frac{-A_t}{A_N}\right)^2 (P_x)^2. \tag{74}$$

In a like manner

$$\left(\frac{\partial A_N}{\partial \dot{y}}\right)^2 = \left(\frac{-A_t}{A_N}\right)^2 (P_y)^2 \tag{75}$$

$$\left(\frac{\partial A_N}{\partial \dot{z}}\right)^2 = \left(\frac{-A_t}{A_N}\right)^2 (P_z)^2 \tag{76}$$

$$\left(\frac{\partial A_N}{\partial \ddot{x}}\right)^2 = \left[\frac{1}{2} (A_s^2 - A_t^2)^{-1/2} \left(2\ddot{x} - \frac{2 A_t \dot{x}}{V_t}\right)\right]^2 = \left(\frac{\ddot{x}}{A_N} - \frac{A_t \dot{x}}{A_N V_t}\right)^2 \tag{77}$$

$$\left(\frac{\partial A_N}{\partial \ddot{y}}\right)^2 = \left(\frac{\ddot{y}}{A_N} - \frac{A_t \dot{y}}{A_N V_t}\right)^2 \tag{78}$$

$$\left(\frac{\partial A_N}{\partial \ddot{z}}\right)^2 = \left(\frac{\ddot{z}}{A_N} - \frac{A_t \dot{z}}{A_N V_t}\right)^2. \tag{79}$$

The variance of the normal acceleration is

$$\sigma_{A_N}^2 = \left(\frac{-A_t}{A_N}\right)^2 \left[(P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2\right] + \left(\frac{V_t \ddot{x} - A_t \dot{x}}{A_N V_t}\right)^2 \sigma_{\ddot{x}}^2$$
$$+ \left(\frac{V_t \ddot{y} - A_t \dot{y}}{A_N V_t}\right)^2 \sigma_{\ddot{y}}^2 + \left(\frac{V_t \ddot{z} - A_t \dot{z}}{A_N V_t}\right)^2 \sigma_{\ddot{z}}^2. \tag{80}$$

Using equations (64), (65) and (66)

$$\sigma^2_{A_N} = \left(\frac{-A_t}{A_N}\right)^2 \left[(P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2\right] + \left(\frac{V_t}{A_N}\right)^2 \left[(P_x)^2 \sigma_{\ddot{x}}^2 + (P_y)^2 \sigma_{\ddot{y}}^2 + (P_z)^2 \sigma_{\ddot{z}}^2\right]. \tag{81}$$

The standard deviation of the normal acceleration is

$$\sigma_{A_N} = \left[\left(\frac{-A_t}{A_N}\right)^2 \left((P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2\right) + \left(\frac{V_t}{A_N}\right)^2 \left((P_x)^2 \sigma_{\ddot{x}}^2 + (P_y)^2 \sigma_{\ddot{y}}^2 + (P_z)^2 \sigma_{\ddot{z}}^2\right)\right]^{\frac{1}{2}}. \tag{82}$$

Rate of change of trajectory angles:

The rate of change of the azimuth angle is

$$\dot{\phi} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^2 + \dot{y}^2}. \tag{34}$$

The variance is given by

$$\sigma^2_{\dot{\phi}} = \left(\frac{\partial\dot{\phi}}{\partial\dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial\dot{\phi}}{\partial\dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial\dot{\phi}}{\partial\ddot{x}}\right)^2 \sigma_{\ddot{x}}^2 + \left(\frac{\partial\dot{\phi}}{\partial\ddot{y}}\right)^2 \sigma_{\ddot{y}}^2 \tag{83}$$

where

$$\left(\frac{\partial\dot{\phi}}{\partial\dot{x}}\right)^2 = \left(\frac{(\dot{x}^2 + \dot{y}^2)\,\ddot{y} - (\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})\,2\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2}\right)^2 = \left(\frac{\ddot{y} - 2\dot{x}\,\dot{\phi}}{\dot{x}^2 + \dot{y}^2}\right)^2 \tag{84}$$

$$\left(\frac{\partial\dot{\phi}}{\partial\dot{y}}\right)^2 = \left(\frac{(\dot{x}^2 + \dot{y}^2)\,\ddot{x} - (\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})\,2\dot{y}}{(\dot{x}^2 + \dot{y}^2)^2}\right)^2 = \left(\frac{\ddot{x} - 2\dot{y}\,\dot{\zeta}}{\dot{x}^2 + \dot{y}^2}\right)^2 \tag{85}$$

$$\left(\frac{\partial\dot{\phi}}{\partial\ddot{x}}\right)^2 = \left(\frac{\dot{y}\,(\dot{x}^2 + \dot{y}^2)}{(\dot{x}^2 + \dot{y}^2)^2}\right)^2 = \left(\frac{\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)^2 \tag{86}$$

$$\left(\frac{\partial\dot{\phi}}{\partial\ddot{y}}\right)^2 = \left(\frac{\dot{x}}{\dot{x}^2 + \dot{y}^2}\right)^2. \tag{87}$$

Substituting in equation (83) yields

$$\sigma_{\dot{\phi}}^{2} = \left(\frac{\ddot{y} - 2\dot{x}\,\dot{\phi}}{\dot{x}^2 + \dot{y}^2}\right)^2 \sigma_{\dot{x}}^{2} + \left(\frac{\ddot{x} - 2\dot{y}\,\dot{\phi}}{\dot{x}^2 + \dot{y}^2}\right)^2 \sigma_{\dot{y}}^{2} + \left(\frac{\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)^2 \sigma_{\ddot{x}}^{2} + \left(\frac{\dot{x}}{\dot{x}^2 + \dot{y}^2}\right)^2 \sigma_{\ddot{y}}^{2} \quad (88)$$

and the standard deviation of the rate of change of the azimuth trajectory angle is

$$\sigma_{\dot{\phi}} = \left[\frac{(\ddot{y} - 2\dot{x}\,\dot{\phi})^2\,\sigma_{\dot{x}}^{2} + (\ddot{x} - 2\dot{y}\,\dot{\phi})^2\,\sigma_{\dot{y}}^{2} + \dot{y}^2\,\sigma_{\ddot{x}}^{2} + \dot{x}^2\,\sigma_{\ddot{y}}^{2}}{(\dot{x}^2 + \dot{y}^2)^2}\right]^{\frac{1}{2}}. \quad (89)$$

The rate of change of the elevation angle is

$$\dot{\theta} = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\ \ddot{z} - \dot{z}(\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y})(\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}}}{V_t^{2}}. \quad (36)$$

From equation (54)

$$\dot{G} = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \quad (54)$$

$$\ddot{G} = \frac{d\dot{G}}{dt} = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}}. \quad (90)$$

Using these equations then equation (36) is

$$\dot{\theta} = \frac{\dot{G}\,\ddot{z} - \dot{z}\,\ddot{G}}{V_t^{2}}. \quad (91)$$

The variance of the rate of change of the elevation angle is

$$\sigma_{\dot{\theta}}^{2} = \left(\frac{\partial\dot{\theta}}{\partial\dot{G}}\right)^2 \sigma_{\dot{G}}^{2} + \left(\frac{\partial\dot{\theta}}{\partial\dot{z}}\right)^2 \sigma_{\dot{z}}^{2} + \left(\frac{\partial\dot{\theta}}{\partial\ddot{G}}\right)^2 \sigma_{\ddot{G}}^{2} + \left(\frac{\partial\dot{\theta}}{\partial\ddot{z}}\right)^2 \sigma_{\ddot{z}}^{2} \quad (92)$$

where

$$\left(\frac{\partial\dot{\theta}}{\partial\dot{G}}\right)^2 = \left[\frac{V_t^{2}\,(\ddot{z}) - (\dot{G}\,\ddot{z} - \dot{z}\,\ddot{G})\,2\dot{G}}{V_t^{4}}\right]^2 = \left(\frac{\ddot{z} - 2\dot{G}\,\dot{\theta}}{V_t^{2}}\right)^2 \quad (93)$$

$$\left(\frac{\partial\dot{\theta}}{\partial\dot{z}}\right)^2 = \left[\frac{V_t^{2}\,(-\ddot{G}) - (\dot{G}\,\ddot{z} - \dot{z}\,\ddot{G})\,2\dot{z}}{V_t^{4}}\right]^2 = \left[\frac{-(\ddot{G} + 2\dot{z}\,\dot{\theta})}{V_t^{2}}\right]^2 \quad (94)$$

$$\left(\frac{\partial\dot{\theta}}{\partial\ddot{G}}\right)^2 = \left[\frac{V_t^{2}\,(-\dot{z})}{V_t^{4}}\right]^2 = \left(\frac{-\dot{z}}{V_t^{2}}\right)^2. \quad (95)$$

$$\left(\frac{\partial \dot{\theta}}{\partial \ddot{z}}\right)^2 = \left(\frac{\dot{G}}{V_t^2}\right)^2. \tag{96}$$

Equation (92) then becomes

$$\sigma_{\dot{\theta}}^2 = \frac{(\ddot{z} - 2\dot{G}\,\dot{\theta})^2\,\sigma_{\dot{G}}^2 + (\ddot{G} + 2\dot{z}\,\dot{\theta})^2\,\sigma_{\dot{z}}^2 + \dot{z}^2\,\sigma_{\ddot{G}}^2 + \dot{G}^2\,\sigma_{\ddot{z}}^2}{V_t^4} \tag{97}$$

where

$$\sigma_{\dot{G}}^2 = \frac{\dot{x}^2\,\sigma_{\dot{x}}^2 + \dot{y}^2\,\sigma_{\dot{y}}^2}{\dot{G}^2} \tag{55}$$

and since

$$\ddot{G} = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} \tag{90}$$

the variance is

$$\sigma_{\ddot{G}}^2 = \left(\frac{\partial \ddot{G}}{\partial \dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial \ddot{G}}{\partial \dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial \ddot{G}}{\partial \ddot{x}}\right)^2 \sigma_{\ddot{x}}^2 + \left(\frac{\partial \ddot{G}}{\partial \ddot{y}}\right)^2 \sigma_{\ddot{y}}^2 \tag{98}$$

where

$$\left(\frac{\partial \ddot{G}}{\partial \dot{x}}\right)^2 = \left[\frac{\dot{G}\,\ddot{x} - (\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y})\,\dot{x}\,(\dot{x}^2 + \dot{y}^2)^{-1/2}}{\dot{G}^2}\right]^2 = \left(\frac{\dot{G}\,\ddot{x} - \dot{x}\,\ddot{G}}{\dot{G}^2}\right)^2 \tag{99}$$

$$\left(\frac{\partial \ddot{G}}{\partial \dot{y}}\right)^2 = \left[\frac{\dot{G}\,\ddot{y} - (\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y})\,\dot{y}\,(\dot{x}^2 + \dot{y}^2)^{-1/2}}{\dot{G}^2}\right]^2 = \left(\frac{\dot{G}\,\ddot{y} - \dot{y}\,\ddot{G}}{\dot{G}^2}\right)^2 \tag{100}$$

$$\left(\frac{\partial \ddot{G}}{\partial \ddot{x}}\right)^2 = \left[\frac{\dot{x}\,(\dot{x}^2 + \dot{y}^2)^{1/2}}{\dot{x}^2 + \dot{y}^2}\right]^2 = \left[\frac{\dot{x}\,\dot{G}}{\dot{G}^2}\right]^2 \tag{101}$$

$$\left(\frac{\partial \ddot{G}}{\partial \ddot{y}}\right)^2 = \left[\frac{\dot{y}\,(\dot{x}^2 + \dot{y}^2)^{1/2}}{\dot{x}^2 + \dot{y}^2}\right]^2 = \left[\frac{\dot{y}\,\dot{G}}{\dot{G}^2}\right]^2. \tag{102}$$

Substituting these equations in equation (98) yields

$$\sigma_{\ddot{G}}^2 = \frac{(\dot{G}\,\ddot{x} - \dot{x}\,\ddot{G})^2\,\sigma_{\dot{x}}^2 + (\dot{G}\,\ddot{y} - \dot{y}\,\ddot{G})^2\,\sigma_{\dot{y}}^2 + \dot{x}^2\,\dot{G}^2\,\sigma_{\ddot{x}}^2 + \dot{y}^2\,\dot{G}^2\,\sigma_{\ddot{y}}^2}{\dot{G}^4}. \tag{103}$$

From equation (97) the standard deviation of the rate of change of the elevation angle is

$$\sigma_{\dot{\theta}} = \left[\frac{(\ddot{z} - 2\dot{G}\,\dot{\theta})^2\,\sigma_{\dot{G}}^2 + (\ddot{G} + 2\dot{z}\,\dot{\theta})^2\,\sigma_{\dot{z}}^2 + \dot{z}^2\,\sigma_{\ddot{G}}^2 + \dot{G}^2\,\sigma_{\ddot{z}}^2}{V_t^4}\right]^{\frac{1}{2}}. \tag{104}$$

Radius of curvature:

$$\rho = \frac{V_t^2}{A_N} = \frac{V_t^2}{(A_s^2 - A_t^2)^{\frac{1}{2}}}. \tag{105}$$

The variance of the radius of curvature is

$$\sigma_\rho^2 = \left(\frac{\partial\rho}{\partial\dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial\rho}{\partial\dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial\rho}{\partial\dot{z}}\right)^2 \sigma_{\dot{z}}^2 + \left(\frac{\partial\rho}{\partial\ddot{x}}\right)^2 \sigma_{\ddot{x}}^2 + \left(\frac{\partial\rho}{\partial\ddot{y}}\right)^2 \sigma_{\ddot{y}}^2$$

$$+ \left(\frac{\partial\rho}{\partial\ddot{z}}\right)^2 \sigma_{\ddot{z}}^2 \tag{106}$$

where

$$\left(\frac{\partial\rho}{\partial\dot{x}}\right)^2 = \left[\frac{A_N\,(2\dot{x}) - \dfrac{V_t^2}{A_N}\,(-A_t)\,\dfrac{\partial A_t}{\partial\dot{x}}}{A_N^2}\right]^2. \tag{107}$$

Using equations (64) and (105) then

$$\left(\frac{\partial\rho}{\partial\dot{x}}\right)^2 = \left(\frac{2\dot{x}\,A_N + \rho\,A_t\,P_x}{A_N^2}\right)^2 \tag{108}$$

and in a similar manner

$$\left(\frac{\partial\rho}{\partial\dot{y}}\right)^2 = \left(\frac{2\dot{y}\,A_N + \rho\,A_t\,P_y}{A_N^2}\right)^2 \tag{109}$$

$$\left(\frac{\partial\rho}{\partial\dot{z}}\right)^2 = \left(\frac{2\dot{z}\,A_N + \rho\,A_t\,P_z}{A_N^2}\right)^2 \tag{110}$$

$$\left(\frac{\partial \rho}{\partial \ddot{x}}\right)^2 = \left\{ \frac{\dfrac{-V_t^2}{A_N} \left[A_s \dfrac{\partial A_s}{\partial \ddot{x}} - A_t \dfrac{\partial A_t}{\partial \ddot{x}} \right]}{A_N^2} \right\}^2 \qquad (111)$$

$$\left(\frac{\partial \rho}{\partial \ddot{x}}\right)^2 = \left[\frac{-\rho\left(\ddot{x} - A_t \dfrac{\dot{x}}{V_t}\right)}{A_N^2} \right]^2 = \left[\frac{\rho\left(\dfrac{\ddot{x}\, V_t - A_t\, \dot{x}}{V_t}\right)}{A_N^2} \right]^2 . \qquad (112)$$

Since

$$\left(\frac{\ddot{x}\, V_t - A_t\, \dot{x}}{V_t^2}\right) = P_x$$

then

$$\left(\frac{\partial \rho}{\partial \ddot{x}}\right)^2 = \left(\frac{\rho\, V_t\, (P_x)}{A_N^2}\right)^2 \qquad (113)$$

and in a like manner

$$\left(\frac{\partial \rho}{\partial \ddot{y}}\right)^2 = \left(\frac{\rho\, V_t\, (P_y)}{A_N^2}\right)^2 \qquad (114)$$

$$\left(\frac{\partial \rho}{\partial \ddot{z}}\right)^2 = \left(\frac{\rho\, V_t\, (P_z)}{A_N^2}\right)^2 . \qquad (115)$$

The variance then becomes

$$\sigma_\rho^2 = \frac{(2\dot{x}\, A_N + \rho\, A_t\, P_x)^2\, \sigma_{\dot{x}}^2 + (2\dot{y}\, A_N + \rho\, A_t\, P_y)^2\, \sigma_{\dot{y}}^2 + (2\dot{z}\, A_N + \rho\, A_t\, P_z)^2\, \sigma_{\dot{z}}^2}{A_N^4}$$

$$+ \frac{\rho^2\, V_t^2\, [(P_x)^2\, \sigma_{\ddot{x}}^2 + (P_y)^2\, \sigma_{\ddot{y}}^2 + (P_z)^2\, \sigma_{\ddot{z}}^2)]}{A_N^4} . \qquad (116)$$

The standard deviation of the radius of curvature is

$$\sigma_\rho = \frac{1}{A_N^2} \Big[(2\dot{x}\, A_N + \rho\, A_t\, P_x)^2\, \sigma_{\dot{x}}^2 + (2\dot{y}\, A_N + \rho\, A_t\, P_y)^2\, \sigma_{\dot{y}}^2 + (2\dot{z}\, A_N + \rho\, A_t\, P_z)^2\, \sigma_{\dot{z}}^2$$

$$+ \rho^2\, V_t^2\, \left[(P_x)^2\, \sigma_{\ddot{x}}^2 + (P_y)^2\, \sigma_{\ddot{y}}^2 + (P_z)^2\, \sigma_{\ddot{z}}^2)\right] \Big]^{\frac{1}{2}} \qquad (117)$$

Rate of turn:

The rate of turn is

$$\dot{\omega} = \frac{V_t}{\rho} \tag{33}$$

or since

$$\rho = \frac{V_t^2}{A_N} \tag{25}$$

then $$\dot{\omega} = \frac{A_N}{V_t}. \tag{118}$$

The variance is

$$\sigma_{\dot{\omega}}^2 = \left(\frac{\partial\dot{\omega}}{\partial\dot{x}}\right)^2 \sigma_{\dot{x}}^2 + \left(\frac{\partial\dot{\omega}}{\partial\dot{y}}\right)^2 \sigma_{\dot{y}}^2 + \left(\frac{\partial\dot{\omega}}{\partial\dot{z}}\right)^2 \sigma_{\dot{z}}^2 + \left(\frac{\partial\dot{\omega}}{\partial\ddot{x}}\right)^2 \sigma_{\ddot{x}}^2 + \left(\frac{\partial\dot{\omega}}{\partial\ddot{y}}\right)^2 \sigma_{\ddot{y}}^2 + \left(\frac{\partial\dot{\omega}}{\partial\ddot{z}}\right)^2 \sigma_{\ddot{z}}^2 \tag{119}$$

where

$$\left(\frac{\partial\dot{\omega}}{\partial\dot{x}}\right)^2 = \left[V_t \frac{\partial A_N}{\partial\dot{x}} - A_N \frac{\partial V_t}{\partial\dot{x}}\right]^2 \left[\frac{1}{V_t^4}\right]. \tag{120}$$

Using equations (74) and (59)

$$\left(\frac{\partial\dot{\omega}}{\partial\dot{x}}\right)^2 = \left[\frac{-\frac{A_t}{A_N} V_t P_x - \frac{A_N \dot{x}}{V_t}}{V_t^2}\right]^2 = \left[-\left(\frac{V_t^2 A_t P_x}{A_N V_t^3} + \frac{A_N^2 \dot{x}}{A_N V_t^3}\right)\right]^2$$

$$= \left[\frac{-(\rho A_t P_x + A_N \dot{x})}{V_t^3}\right]^2 \tag{121}$$

and in the same manner

$$\left(\frac{\partial\dot{\omega}}{\partial\dot{y}}\right)^2 = \left[\frac{-(\rho\ A_t\ P_y + A_N\ \dot{y})}{V_t^{\,3}}\right]^2 \tag{122}$$

$$\left(\frac{\partial\dot{\omega}}{\partial\dot{z}}\right)^2 = \left[\frac{-(\rho\ A_t\ P_z + A_N\ \dot{z})}{V_t^{\,3}}\right]^2 \tag{123}$$

$$\left(\frac{\partial\dot{\omega}}{\partial\ddot{x}}\right)^2 = \left(\frac{V_t\ \dfrac{\partial A_N}{\partial\ddot{x}}}{V_t^{\,2}}\right)^2 \tag{124}$$

then using equations (64) and (77)

$$\left(\frac{\partial\dot{\omega}}{\partial\ddot{x}}\right)^2 = \left[\frac{V_t^{\,2}\left(\dfrac{P_x}{A_N}\right)}{V_t^{\,2}}\right]^2 = \left(\frac{P_x}{A_N}\right)^2 \tag{125}$$

and

$$\left(\frac{\partial\dot{\omega}}{\partial\ddot{y}}\right)^2 = \left(\frac{P_y}{A_N}\right)^2 \tag{126}$$

$$\left(\frac{\partial\dot{\omega}}{\partial\ddot{z}}\right)^2 = \left(\frac{P_z}{A_N}\right)^2. \tag{127}$$

The variance is then

$$\sigma_{\dot{\omega}}^{\,2} = \frac{\{(\rho\ A_t\ P_x + A_N\ \dot{x})^2\ \sigma_{\dot{x}}^{\,2} + (\rho\ A_t\ P_y + A_N\ \dot{y})^2\ \sigma_{\dot{y}}^{\,2} + (\rho\ A_t\ P_z + A_N\ \dot{z})^2\ \sigma_{\dot{z}}^{\,2}\}}{V_t^{\,6}}$$

$$+ \frac{(P_x)^2\ \sigma_{\ddot{x}}^{\,2} + (P_y)^2\ \sigma_{\ddot{y}}^{\,2} + (P_z)^2\ \sigma_{\ddot{z}}^{\,2}}{A_N^{\,2}} \tag{128}$$

and the standard deviation is

$$\sigma_{\dot{\omega}} = \left[\frac{(\rho\ A_t\ P_x + A_N\ \dot{x})^2\ \sigma_{\dot{x}}^{\,2} + (\rho\ A_t\ P_y + A_N\ \dot{y})^2\ \sigma_{\dot{y}}^{\,2} + (\rho\ A_t\ P_z + A_N\ \dot{z})^2\ \sigma_{\dot{z}}^{\,2}}{V_t^{\,6}}\right.$$

$$\left. + \frac{(P_x)^2\ \sigma_{\ddot{x}}^{\,2} + (P_y)^2\ \sigma_{\ddot{y}}^{\,2} + (P_z)^2\ \sigma_{\ddot{z}}^{\,2}}{A_N^{\,2}}\right]^{\frac{1}{2}}. \tag{129}$$

Computational Procedure

Using the velocity and acceleration components obtained from a smoothing routine and the variances of these components compute the following

Velocity:

$$V_t = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}. \tag{1}$$

Trajectory angles:

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) \tag{2}$$

$$\theta = \tan^{-1}\left(\frac{\dot{z}}{\dot{G}}\right) \tag{3}$$

where

$$\dot{G} = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}. \tag{4}$$

Tangential acceleration:

$$A_t = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}}{V_t}. \tag{5}$$

Total acceleration:

$$A_s = (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)^{\frac{1}{2}}. \tag{6}$$

Normal acceleration:

$$A_N = (A_s{}^2 - A_t{}^2)^{\frac{1}{2}}. \tag{7}$$

Rate of change of trajectory angles:

$$\dot{\phi} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{G}^2} \tag{8}$$

$$\dot{\theta} = \frac{\dot{G}\,\ddot{z} - \dot{z}\,\ddot{G}}{V_t{}^2} \tag{9}$$

where

$$\ddot{G} = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}}{\dot{G}}. \tag{10}$$

Radius of curvature:

$$\rho = \frac{V_t^2}{A_N}. \tag{11}$$

Rate of turn:

$$\dot{\omega} = \frac{V_t}{\rho} \tag{12}$$

or if the radius of curvature is not available

$$\dot{\omega} = \frac{A_N}{V_t}. \tag{13}$$

Standard Deviations:

Velocity:

$$\sigma_{V_t} = \left(\frac{\dot{x}^2 \ \sigma_{\dot{x}}^2 + \dot{y}^2 \ \sigma_{\dot{y}}^2 + \dot{z}^2 \ \sigma_{\dot{z}}^2}{V_t^2} \right)^{\frac{1}{2}}. \tag{14}$$

Trajectory angles.

$$\sigma_{\phi} = \left(\frac{\dot{y}^2 \ \sigma_{\dot{x}}^2 + \dot{x}^2 \ \sigma_{\dot{y}}^2}{\dot{G}^4} \right)^{\frac{1}{2}} \tag{15}$$

$$\sigma_{\theta} = \left(\frac{\dot{z}^2 \ \sigma_{\dot{G}}^2 + \dot{G}^2 \sigma_{\dot{z}}^2}{V_t^4} \right)^{\frac{1}{2}} \tag{16}$$

where $\sigma_{\dot{G}}^2 = \dfrac{\dot{x}^2 \ \sigma_{\dot{x}}^2 + \dot{y}^2 \ \sigma_{\dot{y}}^2}{\dot{G}^2}.$ (17)

Tangential acceleration:

$$(P_x)^2 = \left(\frac{V_t \ddot{x} - \dot{x} A_t}{V_t^2}\right)^2 \tag{18}$$

$$(P_y)^2 = \left(\frac{V_t \ddot{y} - \dot{y} A_t}{V_t^2}\right)^2 \tag{19}$$

$$(P_z)^2 = \left(\frac{V_t \ddot{z} - \dot{z} A_t}{V_t^2}\right)^2 \tag{20}$$

$$\sigma A_t = \left[(P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2 + \frac{\dot{x}^2 \sigma_{\ddot{x}}^2 + \dot{y}^2 \sigma_{\ddot{y}}^2 + \dot{z}^2 \sigma_{\ddot{z}}^2}{V_t^2}\right]^{\frac{1}{2}}. \tag{21}$$

Normal acceleration:

$$\sigma_{A_N} = \left[\left(\frac{-A_t}{A_N}\right)^2 \left[(P_x)^2 \sigma_{\dot{x}}^2 + (P_y)^2 \sigma_{\dot{y}}^2 + (P_z)^2 \sigma_{\dot{z}}^2\right] + \left(\frac{V_t}{A_N}\right)^2 \left[(P_x)^2 \sigma_{\ddot{x}}^2 + (P_y)^2 \sigma_{\ddot{y}}^2 + (P_z)^2 \sigma_{\ddot{z}}^2\right]\right]^{\frac{1}{2}}. \tag{22}$$

Rate of change of trajectory angles:

$$\sigma_{\dot{\phi}} = \left[\frac{(\ddot{y} - 2\dot{x}\,\dot{\phi})^2 \sigma_{\dot{x}}^2 + (\ddot{x} - 2\dot{y}\,\dot{\phi})^2 \sigma_{\dot{y}}^2 + \dot{y}^2 \sigma_{\ddot{x}}^2 + \dot{x}^2 \sigma_{\ddot{y}}^2}{\dot{G}^4}\right]^{\frac{1}{2}} \tag{23}$$

$$\sigma_{\dot{\theta}} = \left[\frac{(\ddot{z} - 2\dot{G}\,\dot{\theta})^2 \sigma_{\dot{G}}^2 + (\ddot{G} + 2\dot{z}\,\dot{\theta})^2 \sigma_{\dot{z}}^2 + \dot{z}^2 \sigma_{\ddot{G}}^2 + \dot{G}^2 \sigma_{\ddot{z}}^2}{V_t^4}\right]^{\frac{1}{2}} \tag{24}$$

where $$\sigma_{\ddot{G}}^2 = \frac{(\dot{G}\,\ddot{x} - \dot{x}\,\ddot{G})^2 \sigma_{\dot{x}}^2 + (\dot{G}\,\ddot{y} - \dot{y}\,\ddot{G})^2 \sigma_{\dot{y}}^2 + \dot{x}^2 \dot{G}^2 \sigma_{\ddot{x}}^2 + \dot{y}^2 \dot{G}^2 \sigma_{\ddot{y}}^2}{\dot{G}^4}. \tag{25}$$

Radius of curvature:

$$\sigma_\rho = \frac{1}{A_N^2} \Big[(2\dot{x} A_N + \rho A_t P_x)^2 \sigma_{\dot{x}}^2 + (2\dot{y} A_N + \rho A_t P_y)^2 \sigma_{\dot{y}}^2$$

$$+ (2\dot{z} A_N + \rho A_t P_z)^2 \sigma_{\dot{z}}^2 + \rho^2 V_t^2 \Big((P_x)^2 \sigma_{\ddot{x}}^2 + (P_y)^2 \sigma_{\ddot{y}}^2$$

$$+ (P_z)^2 \sigma_{\ddot{z}}^2 \Big) \Big]^{1/2}. \qquad (26)$$

Rate of turn:

$$\sigma_{\dot{\omega}} = \Bigg[\frac{(\rho A_t P_x + A_N \dot{x})^2 \sigma_{\dot{x}}^2 + (\rho A_t P_y + A_N \dot{y})^2 \sigma_{\dot{y}}^2 + (\rho A_t P_z + A_N \dot{z})^2 \sigma_{\dot{z}}^2}{V_t^6}$$

$$+ \frac{(P_x)^2 \sigma_{\ddot{x}}^2 + (P_y)^2 \sigma_{\ddot{y}}^2 + (P_z)^2 \sigma_{\ddot{z}}^2}{A_N^2} \Bigg]^{1/2}. \qquad (27)$$

D. VELOCITY AND ACCELERATION

IV Angular Velocity and Acceleration

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ANGULAR VELOCITY AND ACCELERATION

Introduction:

This program computes the angular velocity and acceleration (tracking rates) of a given Contraves camera. These tracking rates are inserted in the Contraves Predetermined Function Generator (PFG), which permits automatic tracking of a missile. The PFG directs the Contraves telescope to a missile whose launch position and probable trajectory are known. The Contraves operator has only to make slight corrections with the joy stick. After the predetermined function has been completed the control automatically remains in the hands of the operator. The PFG is used whenever extremely high acceleration does not permit manual tracking or whenever the missile is obscured from view during the first few seconds of flight.

Description of Predetermined Function Generator:

The PFG is, in principle, an electro-mechanical translator. The elevation and azimuth functions are generated by cams the shapes of which produce the functions in polar coordinates. A feeler resting on the periphery of each disc actuates a precision potentiometer as a function of the radial coordinate of the cam. The potentiometer then supplies a voltage proportional to the predetermined function to the theodolite. The cams are driven by a synchronous motor which insures that the time factor will be exactly represented.

The starting pulse after amplification actuates a relay which starts the synchronous motor and simultaneously closes the circuit from the potentiometer to the theodolite. After about 19 seconds a limiting contact stops the synchronous motor and returns the Contravos to the manual tracking mode.

The unit is of simple and sturdy construction. The two function cams are slipped onto the shaft of a turntable calibrated in seconds. The positions of the potentiometer feelers are indicated by dials calibrated in angular speed.

Definitions of symbols:

t_0 = time at beginning of the interval

t_1 = time at end of the interval

α_0 = azimuth angle at time t_0

α_1 = azimuth angle at time t_1

ϵ_0 = elevation angle at time t_0

ϵ_1 = elevation angle at time t_1

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$\dot{\alpha}_o$ = azimuth velocity at time t_o

$\dot{\alpha}_i$ = azimuth velocity at time t_i

$\dot{\varepsilon}_o$ = elevation velocity at time t_o

$\dot{\varepsilon}_i$ = elevation velocity at time t_i

$\ddot{\alpha}_{vi}$ = azimuth acceleration computed from azimuth velocity at time t_i

$\ddot{\varepsilon}_{vi}$ = elevation acceleration computed from elevation velocity at time t_i

$\ddot{\alpha}_{pi}$ = azimuth acceleration computed from azimuth angle at time t_i

$\ddot{\varepsilon}_{pi}$ = elevation acceleration computed from elevation angle at time t_i

x_i, y_i, z_i = coordinates of missile with respect to camera at the i^{th} time

V_{xi}, V_{yi}, V_{zi} = velocity components of missile with respect to camera at the i^{th} time

X_L, Y_L, H_L = WSTM coordinates of launcher

X_C, Y_C, H_C = WSTM coordinates of camera

X_{mi}, Y_{mi}, Z_{mi} = coordinates of probable or standard trajectory at the i^{th} time

V_{xmi}, V_{ymi}, V_{zmi} = velocity components of probable or standard trajectory at the i^{th} time

CF = conversion factor for desired units

QE = quadrant elevation of missile

$\not\triangleleft A$ = azimuth of fire of missile to be launched

$\not\triangleleft E$ = (QE of missile to be fired) - (QE of standard missile)

Mathematical Discussion:

The rates to be inserted into the PFG will be the azimuth and elevation accelerations that best describe the changes which occur in the azimuth and elevation during the time interval t_o thru t_i. The PFG utilizes up to ten of these angular accelerations and time intervals.

The angular accelerations may be computed from either the angular velocity or the position (angles).

Accelerations based on angular velocity are computed using the following equations:

$$\ddot{\alpha}_{vi} = \frac{\dot{\alpha}_i - \dot{\alpha}_o}{t_i - t_o}$$

$$\ddot{\varepsilon}_{vi} = \frac{\dot{\varepsilon}_i - \dot{\varepsilon}_c}{t_i - t_o}$$

Accelerations based on position (angles) are computed from:

$$\ddot{\alpha}_{pi} = \frac{2[\alpha_i - \alpha_o - \dot{\alpha}_o (t_i - t_o)]}{(t_i - t_o)^2}$$

$$\ddot{\varepsilon}_{pi} = \frac{2[\varepsilon_i - \varepsilon_o - \dot{\varepsilon}_c (t_i - t_o)]}{(t_i - t_o)^2}$$

If $\dddot{\alpha}_o = \dddot{\alpha}_i$, then azimuth accelerations computed from the two methods will be equal and the position (angle) and azimuth velocity will be correct at time t_i.

If $\dddot{\alpha}_o \neq \dddot{\alpha}_i$, then computing azimuth acceleration from the velocity will introduce a constant error into the position at time t_i but the azimuth velocity will be correct at t_i.

If $\dddot{\alpha}_o \neq \dddot{\alpha}_i$ and azimuth acceleration is computed from the positions, an error will be introduced in the azimuth velocity but the position at t_i will be correct. However, the error in the azimuth velocity will produce a cumulative error in the succeeding positions. This cumulative error will soon exceed the constant position error introduced by the first method of computation. Therefore the following reduction is based on the first method of computing accelerations from velocities. The previous discussion also applies to the elevation acceleration computation.

The angular velocities are derived as follows:

If $u = F(x, y, z)$

and $x = f_1 (t)$

$y = f_2 (t)$

$z = f_3 (t)$

then $\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)$

or $\dot{u} = \left(\frac{\partial u}{\partial x}\right)V_x + \left(\frac{\partial u}{\partial y}\right)V_y + \left(\frac{\partial u}{\partial z}\right)V_z$

If $\alpha_i = u_i = \tan^{-1}\left(\frac{y_i}{x_i}\right)$

and

$$\dot{\alpha}_i = \dot{u}_i = \left(\frac{\partial \alpha_i}{\partial x}\right)V_{xi} + \left(\frac{\partial \alpha_i}{\partial y}\right)V_{yi} + \left(\frac{\partial \alpha_i}{\partial z}\right)V_{zi} \qquad (1)$$

where

$$\frac{\partial \alpha_i}{\partial x} = \frac{-y_i}{x_i^2 + y_i^2}$$

$$\frac{\partial \alpha_i}{\partial y} = \frac{x_i}{x_i^2 + y_i^2}$$

$$\frac{\partial \alpha_i}{\partial z} = 0$$

Substituting $\frac{\partial \alpha_i}{\partial x}$, $\frac{\partial \alpha_i}{\partial y}$, $\frac{\partial \alpha_i}{\partial z}$ in equation (1), it is easily seen that

$$\dot{\alpha} = \frac{x_i V_{yi} - y_i V_{xi}}{x_i^2 + y_i^2} \qquad (2)$$

If $\varepsilon_i = u_i = \tan^{-1}\left[\frac{z_i}{(x_i^2 + y_i^2)^{\frac{1}{2}}}\right]$

and $\dot{\varepsilon}_i = \dot{u}_i = \left(\frac{\partial \varepsilon_i}{\partial x}\right)V_{xi} + \left(\frac{\partial \varepsilon_i}{\partial y}\right)V_{yi} + \left(\frac{\partial \varepsilon_i}{\partial z}\right)V_{zi}$ (3)

where

$$\frac{\partial \varepsilon_i}{\partial x} = \frac{-x_i \; z_i}{(x_i^2 + y_i^2)^{1/2} \, (x_i^2 + y_i^2 + z_i^2)}$$

$$\frac{\partial \varepsilon_i}{\partial y} = \frac{-y_i \; z_i}{(x_i^2 + y_i^2)^{1/2} \, (x_i^2 + y_i^2 + z_i^2)}$$

$$\frac{\partial \varepsilon_i}{\partial z} = \frac{(x_i^2 + y_i^2)^{1/2}}{(x_i^2 + y_i^2 + z_i^2)}$$

Substituting $\frac{\partial \varepsilon i}{\partial x}$, $\frac{\partial \varepsilon i}{\partial y}$, $\frac{\partial \varepsilon i}{\partial z}$ in equation (3),

$$\dot{\varepsilon} = \frac{(x_i^2 + y_i^2) \, V_{zi} - z_i \, (x_i \, V_{xi} + Y_i \, V_{yi})}{(x_i^2 + y_i^2)^{1/2} \, (x_i^2 + y_i^2 + z_i^2)} \tag{4}$$

Computational Procedure:

Angular velocities and accelerations of a camera may be computed from either a standard trajectory supplied by the contractor or from the trajectory data of a missile which has the same trajectory as that expected from the missile to be fired. The trajectory used must be with respect to the missile line of fire.

For each camera compute the coordinates of the launcher with respect to the camera and rotate the coordinates through angle A as follows:

$$X = (Y_L - Y_C) \cos A + (X_L - X_C) \sin A$$

$$Y = -(Y_L - Y_C) \sin A + (X_L - X_C) \cos A$$

$$Z = H_L - H_C$$

where $\measuredangle$ A = Azimuth of fire of missile to be launched

X_L, Y_L, H_L = WSTM coordinates of launcher

X_C, Y_C, H_C = WSTM coordinates of camera

The probable or standard trajectory data for each i^{th} time required is then rotated through angle E. The equations for rotating positions are:

$$X'_{mi} = X_{mi} \cos E - Z_{mi} \sin E$$

$$Y'_{mi} = Y_{mi}$$

$$Z'_{mi} = X_{mi} \sin E + Z_{mi} \cos E$$

and velocity components become:

$$V_{xi} = V_{xmi} \cos E - V_{zmi} \sin E$$

$$V_{yi} = V_{ymi}$$

$$V_{zi} = V_{xmi} \sin E + V_{zmi} \cos E$$

where

$\measuredangle E$ = (QE of missile to be fired) - (QE of standard trajectory)

X_{mi}, Y_{mi}, Z_{mi} = Coordinates of probable or standard trajectory at the i^{th} time

$V_{xmi}, V_{ymi}, V_{zmi}$ = Velocity components of probable or standard trajectory at the i^{th} time

The positions with respect to the camera at the i^{th} time then become:

$$x_i = (X)CF + X'_{mi}$$

$$y_i = (Y)CF + Y'_{mi}$$

$$z_i = (Z)CF + Z'_{mi}$$

where CF = conversion factor to desired units.

The angular velocities are computed as follows:

$$\dot{\alpha}_i = \frac{x_i V_{y_i} - y_i V_{x_i}}{x_i^2 + y_i^2}$$

$$\dot{\varepsilon}_i = \frac{(x_i^2 + y_i^2) V_{z_i} - z_i (x_i V_{x_i} + y_i V_{y_i})}{(x_i^2 + y_i^2)^{1/2} (x_i^2 + y_i^2 + z_i^2)}$$

and the angular accelerations, computed from the angular velocities, are:

$$\ddot{\alpha}_{v_i} = \frac{\dot{\alpha}_i - \dot{\alpha}_{(i-1)}}{t_i - t_{(i-1)}}$$

$$\ddot{\varepsilon}_{v_i} = \frac{\dot{\varepsilon}_i - \dot{\varepsilon}_{(i-1)}}{t_i - t_{(i-1)}}$$

$\dot{\alpha}_i$, $\dot{\varepsilon}_i$ are in degrees/sec

$\ddot{\alpha}_{v_i}$, $\ddot{\varepsilon}_{v_i}$ are in degrees/sec^2

Addendum to the Computational Procedure of the Angular Velocity and Acceleration Report in the Handbook.

Another method of computing the position of the missile with respect to the camera is to rotate the position of the missile at the i^{th} time through the angle, E, then through the angle A, and then translate this position so that it will be with respect to the camera. The rotations may be shown in matrix form as

$$\begin{bmatrix} \cos A & -\sin A & 0 \\ \sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos E & 0 & -\sin E \\ 0 & 1 & 0 \\ \sin E & 0 & \cos E \end{bmatrix} \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = \begin{bmatrix} X'_{mi} \\ Y'_{mi} \\ Z'_{mi} \end{bmatrix} \qquad (1)$$

or

$$[A] \quad [E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = \begin{bmatrix} X'_{mi} \\ Y'_{mi} \\ Z'_{mi} \end{bmatrix} \qquad (2)$$

Solving for X'_{mi} , Y'_{mi} , Z'_{mi} yields

$$\begin{aligned} X'_{mi} &= \cos A\,(X_{mi} \cos E - Z_{mi} \sin E) - Y_{mi} \sin A \\ Y'_{mi} &= \sin A\,(X_{mi} \cos E - Z_{mi} \sin E) + Y_{mi} \cos A \qquad (3) \\ Z'_{mi} &= X_{mi} \sin E + Z_{mi} \cos E \end{aligned}$$

The missile position with respect to the camera may then be obtained from

$$\begin{aligned} X'_i &= X'_{mi} + (Y_L - YC) \\ Y'_i &= Y'_{mi} + (X_L - X_C) \qquad (4) \\ Z'_i &= Z'_{mi} + (H_L - H_C) \end{aligned}$$

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or using equation (2) the rotation and translation become

$$\begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} + [A]\ [E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = \begin{bmatrix} X'_{mi} \\ Y'_{mi} \\ Z'_{mi} \end{bmatrix} + \begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} = \begin{bmatrix} X'_i \\ Y'_i \\ Z'_i \end{bmatrix} \qquad (5)$$

If the positions are rotated and translated in the above manner then the velocity components must also be rotated through the same angles as the positions were. That is,

$$[A] \quad [E] \begin{bmatrix} V_{xmi} \\ V_{ymi} \\ V_{zmi} \end{bmatrix} = \begin{bmatrix} V'_{xi} \\ V'_{yi} \\ V'_{zi} \end{bmatrix} \qquad (6)$$

The results of equations (5) and (6) are then used in the equations for finding the angular velocities and accelerations.

It may be of interest, at this point, to note that equation (5) above and the equations for X_i , Y_i Z_i in the Handbook are not equal. This is explained in the following paragraphs.

In the Handbook the coordinates of the launcher with respect to the camera are first rotated through angle A by use of the inverse matrix $[A]^{-1}$.

$$[A]^{-1} \begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad (7)$$

Since [A] is an orthogonal transformation we have

$$[A]^{-1} = [A]' = \begin{bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

and solving for X, Y, Z in equation (7) yields the equations in the Handbook. That is,

$$\begin{aligned} X &= (X_L - X_C) \sin A + (Y_L - Y_C) \cos A \\ Y &= (X_L - X_C) \cos A - (Y_L - Y_C) \sin A \\ Z &= H_L - H_C \end{aligned} \quad (9)$$

The next step in the Handbook is to rotate the position of the missile at the ith time through the angle E. In matrix from this becomes

$$[E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = \begin{bmatrix} X'_{mi} \\ Y'_{mi} \\ Z'_{mi} \end{bmatrix}' \quad (10)$$

Equations (7) and (10) are then added to obtain the X_i, Y_i, Z_i listed in the Handbook.

$$[A]^{-1} \begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} + [E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \quad (11)$$

If it is desired to obtain the position of the missile with respect to the camera in the same reference system as in equation (5) then one more rotation through the angle Λ is necessary. Rotating equation (11) through angle Λ yields the following matrix form:

$$[\Lambda] \quad [\Lambda]^{-1} \begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} + [\Lambda] \; [E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} = [\Lambda] \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} = \begin{bmatrix} X_i' \\ Y_i' \\ Z_i' \end{bmatrix} \qquad (12)$$

or

$$\begin{bmatrix} Y_L - Y_C \\ X_L - X_C \\ H_L - H_C \end{bmatrix} + [\Lambda] \; [E] \begin{bmatrix} X_{mi} \\ Y_{mi} \\ Z_{mi} \end{bmatrix} \qquad \begin{bmatrix} X_i' \\ Y_i' \\ Z_i' \end{bmatrix} \qquad (13)$$

and equation (13) is equal to equation (5).

This last rotation is unnecessary because the only things being considered in this reduction are the angular velocities and accelerations. If the azimuth angle, , were to be computed from equation (5) it would differ from the azimuth angle computed from equation (11) only by the constant angle Λ. Computing the angular velocity using the azimuth angle in the case of that computed from equation (5) yields

$$\frac{d(\alpha)}{dt} = \dot{\alpha}$$

and using equation (11) gives

$$\frac{d(\alpha - \Lambda)}{dt} = \dot{\alpha} - 0 = \dot{\alpha}$$

D. VELOCITY AND ACCELERATION

V Positional Derivatives from Range or Angular Derivatives

POSITIONAL DERIVATIVES FROM RANGE OR ANGULAR DERIVATIVES

Introduction:

This is a method used to compute the positional velocity and acceleration components using derivative data obtained from ranging or angular measuring systems.

The least squares method is employed to obtain the positional velocity and acceleration components.

The equations used to compute the velocity and acceleration components and their variances are derived in this report.

Several applications of this technique are shown.

Mathematical Procedure:

Velocity Components

The coordinates (x, y, z) of a point, P, on a curve are expressed as functions of a third variable, or parameter, t, in the form

$$u = F(x, y, z)$$

where

$$x = f_1(t)$$
$$y = f_2(t) \qquad (1)$$
$$z = f_3(t).$$

The velocity, or time rate of change of the moving point, P, at any instant is found by taking the derivative of the function, u, with respect to t.

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{dz}{dt}\right) \qquad (2)$$

where $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ are the positional velocity components.

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dy}{dt} = \dot{y}$$

$$\frac{dz}{dt} = \dot{z}.$$

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Letting

$$\frac{\partial u_j}{\partial x} = a_j$$

$$\frac{\partial u_j}{\partial y} = b_j \tag{3}$$

$$\frac{\partial u_j}{\partial z} = c_j$$

and substituting in equation (2) gives

$$\frac{du_j}{dt} = \dot{u}_j = \dot{x}\, a_j + \dot{y}\, b_j + \dot{z}\, c_j \tag{4}$$

for the j^{th} observation, $j = 1, 2, 3 \cdots n$.

Using the least squares procedure on equation (4), the sum of the squares of the residuals is given by

$$S = \sum_{j=1}^{n} [\dot{u}_j - \dot{x}\, a_j - \dot{y}\, b_j - \dot{z}\, c_j]^2. \tag{5}$$

The sum, S, is minimized by equating its partial derivatives, $\frac{\partial S}{\partial \dot{x}}$, $\frac{\partial S}{\partial \dot{y}}$, and $\frac{\partial S}{\partial \dot{z}}$, to zero and solving the three simultaneous equations for $\dot{x}$, $\dot{y}$, and $\dot{z}$. These equations are:

$$\dot{x}\, \Sigma(a_j)^2 + \dot{y}\, \Sigma(a_j\, b_j) + \dot{z}\, \Sigma(a_j\, c_j) = \Sigma(\dot{u}_j\, a_j)$$

$$\dot{x}\, \Sigma(a_j\, b_j) + \dot{y}\, \Sigma(b_j)^2 + \dot{z}\, \Sigma(b_j\, c_j) = \Sigma(\dot{u}_j\, b_j) \tag{6}$$

$$\dot{x}\, \Sigma(a_j\, c_j) + \dot{y}\, \Sigma(b_j\, c_j) + \dot{z}\, \Sigma(c_j)^2 = \Sigma(\dot{u}_j\, c_j).$$

In matrix form these equations become

$$\begin{bmatrix} \Sigma(a_j)^2 & \Sigma(a_j\ b_j) & \Sigma(a_j\ c_j) \\ \Sigma(a_j\ b_j) & \Sigma(b_j)^2 & \Sigma(b_j\ c_j) \\ \Sigma(a_j\ c_j) & \Sigma(b_j\ c_j) & \Sigma(c_j)^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \Sigma(\dot{u}_j\ a_j) \\ \Sigma(\dot{u}_j\ b_j) \\ \Sigma(\dot{u}_j\ c_j) \end{bmatrix} . \tag{7}$$

$\dot{x}$, $\dot{y}$ and $\dot{z}$ may then be solved for using the following matrix form.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \Sigma(a_j)^2 & \Sigma(a_j\ b_j) & \Sigma(a_j\ c_j) \\ \Sigma(a_j\ b_j) & \Sigma(b_j)^2 & \Sigma(b_j\ c_j) \\ \Sigma(a_j\ c_j) & \Sigma(b_j\ c_j) & \Sigma(c_j)^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma(\dot{u}_j\ a_j) \\ \Sigma(\dot{u}_j\ b_j) \\ \Sigma(\dot{u}_j\ c_j) \end{bmatrix} . \tag{8}$$

Variances of the Velocity Components

From equation (4), the residual of the j^{th} observation is

$$\delta\dot{u}_j = (\dot{u}_j - \dot{x}\ a_j - \dot{y}\ b_j - \dot{z}\ c_j). \tag{9}$$

An estimate of the variance of $\dot{u}$ is defined as the sum of the residuals squared divided by the degrees of freedom.

$$\sigma_{\dot{u}}^2 = \frac{\Sigma(\delta\dot{u}_j)^2}{\text{d.f.}} \tag{10}$$

where d.f. = n - 3, and n is the total number of observations.

The variances of the velocity components are computed using the $\sigma_{\dot{u}}^2$ above and the elements of the inverse of the least squares coefficient matrix. Rewriting equation (8) as

$$[V] = [A]^{-1}\ [B] \tag{11}$$

then the variance of [V] is $\sigma_{\dot{u}}^2 [A]^{-1}$. If A_{ij} is the element of the i^{th} row and j^{th} column of $[A]^{-1}$ then the variances of the velocity components are

$$\sigma_{\dot{x}}^2 = \sigma_{\dot{u}}^2 A_{11}$$
$$\sigma_{\dot{y}}^2 = \sigma_{\dot{u}}^2 A_{22} \qquad (12)$$
$$\sigma_{\dot{z}}^2 = \sigma_{\dot{u}}^2 A_{33} .$$

Acceleration Components

The acceleration, or time rate of change of the velocity, of the point, P, at any instant is found by taking the derivative of the velocity of the function with respect to t. From equation (4), the velocity of the function is

$$\dot{u}_j = \dot{x}\, a_j + \dot{y}\, b_j + \dot{z}\, c_j. \qquad (4)$$

The derivative of $\dot{u}_j$ with respect to t is

$$\frac{d\dot{u}_j}{dt} = \ddot{u}_j = \dot{x}\frac{da_j}{dt} + a_j\frac{d\dot{x}}{dt} + \dot{y}\frac{db_j}{dt} + b_j\frac{d\dot{y}}{dt} + \dot{z}\frac{dc_j}{dt} + c_j\frac{d\dot{z}}{dt}. \qquad (13)$$

$\frac{d\dot{x}}{dt}$, $\frac{d\dot{y}}{dt}$, and $\frac{d\dot{z}}{dt}$ are the acceleration components and

$$\frac{d\dot{x}}{dt} = \ddot{x}$$
$$\frac{d\dot{y}}{dt} = \ddot{y} \qquad (14)$$
$$\frac{d\dot{z}}{dt} = \ddot{z}$$

and

$$\frac{da_j}{dt} = \dot{a}_j$$
$$\frac{db_j}{dt} = \dot{b}_j \qquad (15)$$
$$\frac{dc_j}{dt} = \dot{c}_j .$$

Substituting equations (14) and (15) in equation (13),

$$\ddot{u}_j = \dot{x}\,\dot{a}_j + \dot{y}\,\dot{b}_j + \dot{z}\,\dot{c}_j + \ddot{x}\,a_j + \ddot{y}\,b_j + \ddot{z}\,c_j . \tag{16}$$

Using the least squares procedure on equation (16), the sum of the squares of the residuals is given by

$$S = \sum_{j=1}^{n} [\ddot{u}_j - \dot{x}\,\dot{a}_j - \dot{y}\,\dot{b}_j - \dot{z}\,\dot{c}_j - \ddot{x}\,a_j - \ddot{y}\,b_j - \ddot{z}\,c_j]^2 . \tag{17}$$

The sum, S, is minimized by equating its partial derivatives $\frac{\partial S}{\partial \ddot{x}}$, $\frac{\partial S}{\partial \ddot{y}}$ and $\frac{\partial S}{\partial \ddot{z}}$, to zero and solving the three simultaneous equations for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$.

These equations are:

$$\begin{aligned}
\ddot{x}\,\Sigma(a_j)^2 + \ddot{y}\,\Sigma(a_j\,b_j) + \ddot{z}\,\Sigma(a_j\,c_j) &= \Sigma[\ddot{u}_j - \ddot{x}\,a_j - \ddot{y}\,b_j - \ddot{z}\,c_j]\,a_j \\
\ddot{x}\,\Sigma(a_j\,b_j) + \ddot{y}\,\Sigma(b_j)^2 + \ddot{z}\,\Sigma(b_j\,c_j) &= \Sigma[\ddot{u}_j - \ddot{x}\,a_j - \ddot{y}\,b_j - \ddot{z}\,c_j]\,b_j \\
\ddot{x}\,\Sigma(a_j\,c_j) + \ddot{y}\,\Sigma(b_j\,c_j) + \ddot{z}\,\Sigma(c_j)^2 &= \Sigma[\ddot{u}_j - \ddot{x}\,a_j - \ddot{y}\,b_j - \ddot{z}\,c_j]\,c_j .
\end{aligned} \tag{18}$$

In matrix form these equations become

$$\begin{bmatrix} \Sigma(a_j)^2 & \Sigma(a_j\,b_j) & \Sigma(a_j\,c_j) \\ \Sigma(a_j\,b_j) & \Sigma(b_j)^2 & \Sigma(b_j\,c_j) \\ \Sigma(a_j\,c_j) & \Sigma(b_j\,c_j) & \Sigma(c_j)^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \Sigma(\ddot{u}_j - \dot{x}\,\dot{a}_j - \dot{y}\,\dot{b}_j - \dot{z}\,\dot{c}_j)\,a_j \\ \Sigma(\ddot{u}_j - \dot{x}\,\dot{a}_j - \dot{y}\,\dot{b}_j - \dot{z}\,\dot{c}_j)\,b_j \\ \Sigma(\ddot{u}_j - \dot{x}\,\dot{a}_j - \dot{y}\,\dot{b}_j - \dot{z}\,\dot{c}_j)\,c_j \end{bmatrix} . \tag{19}$$

Rewriting equation (19) in the form

$$[A]\;[\hat{V}] = [C], \tag{20}$$

$\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ are solved for by using the matrix form

$$[\hat{V}] = [A]^{-1}\;[C]. \tag{21}$$

Variances of the Acceleration Components

Using equation (16), the residual of the j^{th} observation is found to be

$$\delta\ddot{u}_j = [\ddot{u}_j - \dot{x}\,\dot{a}_j - \dot{y}\,\dot{b}_j - \dot{z}\,\dot{c}_j - \ddot{x}\,a_j - \ddot{y}\,b_j - \ddot{z}\,c_j]. \qquad (22)$$

An estimate of the variance of $\ddot{u}$ is defined as the sum of the squares of the residuals divided by the degrees of freedom.

$$\sigma_{\ddot{u}}^2 = \frac{\Sigma(\delta\ddot{u}_j)^2}{d.f.} \qquad (23)$$

where d.f. = n - 3, and n is the total number of observations.

The variances of the acceleration components are found using $\sigma_{\ddot{u}}^2$ above and the elements of the inverse of the least squares coefficient matrix. This matrix is the same as the inverse of the coefficient matrix used to compute the velocities. Using equation (21), the variance of $[\dot{V}]$ is $\sigma_{\ddot{u}}^2\,[A]^{-1}$. A_{ij} is again the element of the i^{th} row and j^{th} column of $[A]^{-1}$ and

$$\begin{aligned} \sigma_{\ddot{x}}^2 &= \sigma_{\ddot{u}}^2\,A_{11} \\ \sigma_{\ddot{y}}^2 &= \sigma_{\ddot{u}}^2\,A_{22} \\ \sigma_{\ddot{z}}^2 &= \sigma_{\ddot{u}}^2\,A_{33}\,. \end{aligned} \qquad (24)$$

Applications

Dovap data: This system measures an ellipsoid of revolution about a preselected transmitter and some i^{th} receiver. Velocity components, using position data from any source and the time rate of change of the loop range from dovap, are found as follows.

The loop range is

$$u_j = [x_t^2 + y_t^2 + z_t^2]^{\frac{1}{2}} + [x_i^2 + y_i^2 + z_i^2]^{\frac{1}{2}}$$

where

x_t, y_t, z_t are position coordinates of the missile with respect to the transmitter at time t,

x_i, y_i z_i are position coordinates of the missile with respect to the i^{th} receiver at time t.

Then

$$a_j = \left(\frac{\partial u_j}{\partial x}\right) = \left(\frac{x_t}{R_t} + \frac{x_i}{R_i}\right)$$

$$b_j = \left(\frac{\partial u_j}{\partial y}\right) = \left(\frac{y_t}{R_t} + \frac{y_i}{R_i}\right)$$

$$c_j = \left(\frac{\partial u_j}{\partial z}\right) = \left(\frac{z_t}{R_t} + \frac{z_i}{R_i}\right)$$

where

$$R_t = [x_t^2 + y_t^2 + z_t^2]^{\frac{1}{2}}$$

$$R_i = [x_i^2 + y_i^2 + z_i^2]^{\frac{1}{2}}$$

$\dot{u}_j$ is the time rate of change, or velocity, of the loop range observed from the j^{th} dovap at time t.

a_j, b_j, c_j and $\dot{u}_j$ are computed for each station and substituted in equation (8) to solve for $\dot{x}$, $\dot{y}$ and $\dot{z}$.

Acceleration components are found in a similar manner using position and velocity data from any source and the time rate of change of the velocity of the loop range. a_j, b_j and c_j are the same as those computed for velocities and

$$\dot{a}_j = \frac{da_j}{dt} = \left(\frac{R_t \dot{x} - x_t \dot{R}_t}{R_t^2}\right) + \left(\frac{R_i \dot{x} - x_i \dot{R}_i}{R_i^2}\right)$$

$$\dot{b}_j = \frac{db_j}{dt} = \left(\frac{R_t \dot{y} - y_t \dot{R}_t}{R_t^2}\right) + \left(\frac{R_i \dot{y} - y_i \dot{R}_i}{R_i^2}\right)$$

$$\dot{c}_j = \frac{dc_j}{dt} = \left(\frac{R_t \dot{z} - z_t \dot{R}_t}{R_t^2}\right) + \left(\frac{R_i \dot{z} - z_i \dot{R}_i}{R_i^2}\right)$$

where

$$R_t = [x_t^2 + y_t^2 + z_t^2]^{\frac{1}{2}}$$

$$\dot{R}_t = \frac{x_t \dot{x} + y_t \dot{y} + z_t \dot{z}}{R_t}$$

and

$$R_i = [x_j^2 + y_i^2 + z_1^2]^{\frac{1}{2}}$$

$$\dot{R}_i = \frac{x_i \dot{x} + y_i \dot{y} + z_i z}{R_i}$$

$\ddot{u}_j$ is the time rate of change of the velocity of the loop range.

Equation (19) may then be used to solve for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$.

Cinetheodolite data: This system measures a line in space defined by an azimuth angle (α_j) and an elevation angle (ε_j). Velocity components, using position from any source and the time rate of change of the azimuth and elevation angles, are found in the following manner. In this case each azimuth angle is considered as an individual observation and each elevation angle is considered as an individual observation. If only azimuth angles are available there is no solution for the velocity component in the z direction.

When

$$u_j = \alpha_j = \tan^{-1}\left(\frac{y_j}{x_j}\right),$$

$$a_j = \frac{\partial u_j}{\partial x} = \frac{-y_j}{x_j^2 + y_j^2}$$

$$b_j = \frac{\partial u_j}{\partial y} = \frac{x_j}{x_j^2 + y_j^2}$$

$$c_j = \frac{\partial u_j}{\partial z} = 0 .$$

When

$$u_j = \varepsilon_j = \tan^{-1} \frac{z_j}{(x_j^2 + y_j^2)^{1/2}} ,$$

$$a_j = \frac{\partial u_j}{\partial x} = \frac{- x_j \; z_j}{(x_j^2 + y_j^2)^{1/2} \; (x_j^2 + y_j^2 + z_j^2)}$$

$$b_j = \frac{\partial u_j}{\partial y} = \frac{- y_j \; z_j}{(x_j^2 + y_j^2)^{1/2} \; (x_j^2 + y_j^2 + z_j^2)}$$

$$c_j = \frac{\partial u_j}{\partial z} = \frac{(x_j^2 + y_j^2)^{1/2}}{x_j^2 + y_j^2 + z_j^2} .$$

In both cases x_j, y_j, z_j are position coordinates with respect to the j^{th} theodolite at time t. This process is followed for each azimuth and elevation of each theodolite at the time t. The cofactors of the least squares coefficient matrix are computed and from equation (8) $\dot{x}$, $\dot{y}$, $\dot{z}$ are then found. $\dot{u}_j$ is the rate of change of the angle.

The same process is used to compute acceleration components. The cofactors of the least squares matrix remain the same as those used to compute velocity components. When

$$u_j = \alpha_j$$

then

$$\dot{a}_j = \frac{da_j}{dt} = \frac{-\dot{y}\,(x_j^2 + y_j^2) + y_j\,(2x_j\,\dot{x} + 2y_j\,\dot{y})}{(x_j^2 + y_j^2)^2}$$

$$\dot{b}_j = \frac{db_j}{dt} = \frac{\dot{x}\,(x_j^2 + y_j^2) - x_j\,(2x_j\,\dot{x} + 2y_j\,\dot{y})}{(x_j^2 + y_j^2)^2}$$

$$\dot{c}_j = \frac{dc_j}{dt} = 0\ .$$

When

$$u_j = \varepsilon_j$$

then

$$a_j = \frac{-\,x_j\,z_j}{(x_j^2 + y_j^2 + z_j^2)(x_j^2 + y_j^2)^{\frac{1}{2}}}$$

$$b_j = \frac{-\,y_j\,z_j}{(x_j^2 + y_j^2 + z_j^2)(x_j^2 + y_j^2)^{\frac{1}{2}}}$$

$$c_j = \frac{(x_j^2 + y_j^2)^{\frac{1}{2}}}{(x_j^2 + y_j^2 + z_j^2)}\ .$$

If we let

$$D_j = (x_j^2 + y_j^2 + z_j^2)$$

then

$$a_j = \frac{-\,x_j\,z_j}{D_j\,(x_j^2 + y_j^2)^{\frac{1}{2}}}$$

$$b_j = \frac{-y_j z_j}{D_j (x_j^2 + y_j^2)^{\frac{1}{2}}}$$

$$c_j = \frac{(x_j^2 + y_j^2)^{\frac{1}{2}}}{D_j} .$$

Taking the derivatives of a_j, b_j and c_j with respect to t, gives

$$\dot{a}_j = \frac{da_j}{dt} = \frac{-D_j (x_j^2 + y_j^2)(x_j \dot{z} + z_j \dot{x})}{D_j^2 (x_j^2 + y_j^2)^{\frac{3}{2}}} +$$

$$\frac{x_j z_j [D_j (x_j \dot{x} + y_j \dot{y}) + (x_j^2 + y_j^2) \dot{D}_j]}{D_j^2 (x_j^2 + y_j^2)^{\frac{3}{2}}}$$

$$\dot{b}_j = \frac{db_j}{dt} = \frac{-D_j (x_j^2 + y_j^2)(y_j \dot{z} + z_j \dot{y})}{D_j^2 (x_j^2 + y_j^2)^{\frac{3}{2}}} +$$

$$\frac{y_j z_j [D_j (x_j \dot{x} + y_j \dot{y}) + (x_j^2 + y_j^2) \dot{D}_j]}{D_j^2 (x_j^2 + y_j^2)^{\frac{3}{2}}}$$

$$\dot{c}_j = \frac{dc_j}{dt} = \frac{D_j (x_j \dot{x} + y_j \dot{y}) - (x_j^2 + y_j^2) \dot{D}_j}{D_j^2 (x_j^2 + y_j^2)^{\frac{1}{2}}}$$

where

$$\frac{dD_j}{dt} = \dot{D}_j = 2 x_j \dot{x} + 2y_j \dot{y} + 2 z_j \dot{z}.$$

$\ddot{u}_j$ is the time rate of change of the velocity of the angle.

These values are computed for each angle of each theodolite at time t. Equation (19) is then used to solve for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$.

Velocimeter data: This system is a high frequency doppler system which provides the time rate of change of the range.

$$u_j = R_j = (x_j^2 + y_j^2 + z_j^2)^{\frac{1}{2}}$$

and

$$a_j = \frac{\partial u_j}{\partial x} = \frac{x_j}{R_j}$$

$$b_j = \frac{\partial u_j}{\partial y} = \frac{y_j}{R_j}$$

$$c_j = \frac{\partial u_j}{\partial z} = \frac{z_j}{R_j}$$

where x_j, y_j, z_j are position coordinates, from any source, with respect to the velocimeter.

These values are substituted in equation (8) and $\dot{x}$, $\dot{y}$, $\dot{z}$ are computed.

To compute acceleration data, a_j, b_j and c_j are the same values as computed for the velocities and

$$\dot{a}_j = \frac{da_j}{dt} = \frac{d}{dt}\left(\frac{x_j}{R_j}\right) = \frac{R_j \dot{x} - x_j \dot{R}_j}{R_j^2}$$

$$\dot{b}_j = \frac{db_j}{dt} = \frac{d}{dt}\left(\frac{y_j}{R_j}\right) = \frac{R_j \dot{y} - y_j \dot{R}_j}{R_j^2}$$

$$\dot{c}_j = \frac{dc_j}{dt} = \frac{d}{dt}\left(\frac{z_j}{R_j}\right) = \frac{R_j \dot{z} - z_j \dot{R}_j}{R_j^2}$$

where $$\dot{R}_j = \left(\frac{x_j \dot{x} + y_j \dot{y} + z_j \dot{z}}{R_j}\right).$$

$\dot{u}$ is the time rate of change of the velocity of the range. Equation (19) is then used to solve for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$.

E. ROTATIONS AND TRANSLATIONS

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E. ROTATIONS AND TRANSLATIONS

I Earth Centered Inertial Coordinate System

EARTH CENTERED INERTIAL COORDINATE SYSTEM

In this discussion we shall develope and derive the equations needed for transforming component positions, component velocities, and component accelerations in White Sands Cartesian System to component positions, component velocities and component accelerations in an Earth Centered Inertial System.

WSCS is a left-handed system, (x, y, z), with the x and y axes in a plane tangent to the earth. In the following development the origin of the WSCS is translated to the point of tangency of this plane on the Clarke spheroid of 1866. The x-axis is aligned positive to the north; the y-axis positive to the east; and the z-axis positive up along a plumb line at the point of tangency.

The ECI system, (R_x, R_y, R_z), is a right-handed cartesian system with its origin fixed on the earth's spin axis in the equatorial plane. The orientation of its axes remains fixed in space while its origin moves in a path through space coincident with the earth's motion. The initial orientation of the ECI axes is determined for each mission as follows: the positive R_x axis lies in the earth's equatorial plane directed toward the longitudinal meridian of the missile launcher at missile flight time $t = t_0$; the positive R_z axis is directed south along the earth's spin-axis; the R_y axis completes the right-handed set.

Definitions used in the derivation:

- a is the semi-major axis; and b is the semi-minor axis of the Clarke Spheroid of 1866.
- ϕ is the geodetic latitude of the WSCS point of tangency.
- θ is the geocentric latitude of the WSCS point of tangency.
- ϕ is related to θ by $\theta = \tan^{-1}\left(\frac{b^2}{a^2} \tan \phi\right)$.
- λ_L is the longitude of the launcher. (Negative in the Western hemisphere.)
- λ_C is the longitude of the WSCS point of tangency. (253°40' or -106°20').
- ω is the earth's angular velocity in radians/sec. ($+7.29211 \times 10^{-5}$).
- t is the time relative to lift at which a coordinate point in WSCS is to be transformed into ECI coordinates.
- $\Delta\lambda = (\lambda_C + \omega t) - \lambda_L$.
- R is the geocentric radius of the WSCS point of tangency defined by

$$R = \frac{a}{(1 + K \sin^2 \theta)^{\frac{1}{2}}}$$

where

$$K = \frac{a^2 - b^2}{b^2}.$$

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Procedure:

Pos[illegible]

The WSCS is first rotated about the east-west or y-axis thru an angle of $(90 - \phi)$ and yields the coordinate values (x', y', z'). (See Figure 1).

$$x' = x \cos (90 - \phi) - z \sin (90 - \phi)$$

$$y' = y \tag{1}$$

$$z' = x \sin (90 - \phi) + z \cos (90 - \phi)$$

or

$$x' = x \sin \phi - z \cos \phi$$

$$y' = y \tag{2}$$

$$z' = x \cos \phi + z \sin \phi$$

In matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \sin \phi & 0 & -\cos \phi \\ 0 & 1 & 0 \\ \cos \phi & 0 & \sin \phi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3}$$

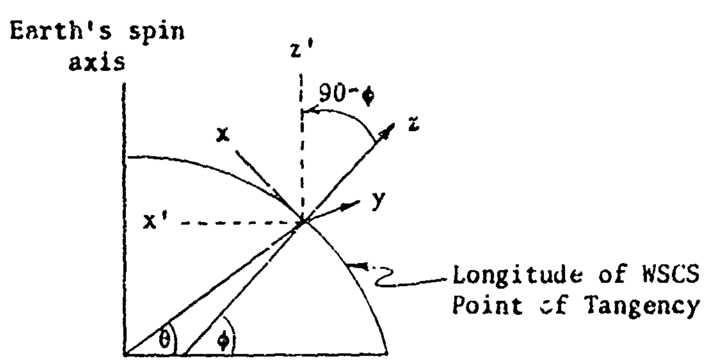

FIGURE 1.

A second rotation about the z' axis thru an angle of $(180° + \Delta\lambda)$ yields coordinate values (x'', y'', z'') (Figure 2).

$$
\begin{aligned}
x'' &= x' \cos (180° + \Delta\lambda) + y' \sin (180° + \Delta\lambda) \\
y'' &= -x' \sin (180° + \Delta\lambda) + y' \cos (180° + \Delta\lambda) \\
z'' &= z'
\end{aligned}
\tag{4}
$$

or

$$
\begin{aligned}
x'' &= -x' \cos \Delta\lambda - y' \sin \Delta\lambda \\
y'' &= x' \sin \Delta\lambda - y' \cos \Delta\lambda \\
z'' &= z'
\end{aligned}
\tag{5}
$$

and in matrix notation

$$
\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} -\cos \Delta\lambda & -\sin \Delta\lambda & 0 \\ \sin \Delta\lambda & -\cos \Delta\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \tag{6}
$$

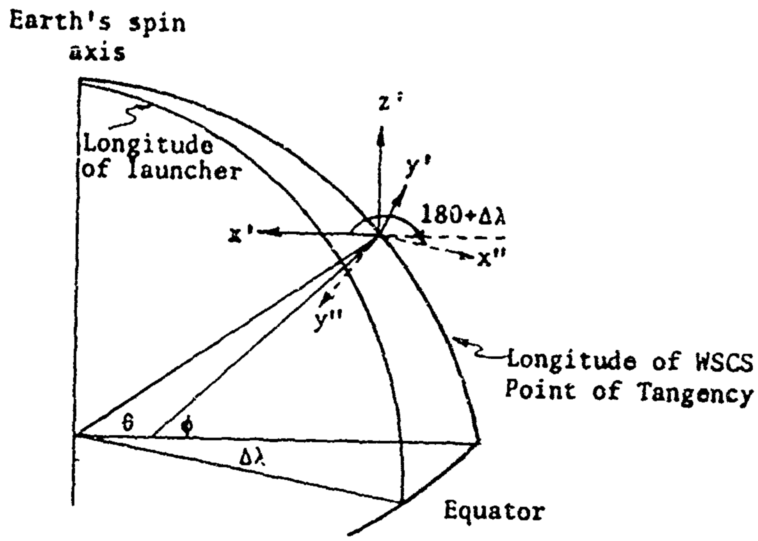

FIGURE 2.

The system (x'', y'', z'') is a left-handed cartesian system which has its axes parallel to the ECI system required. To convert the (x'', y'', z'') to a right-handed cartesian system (x''', y''', z''') with its axes parallel to, and in the same direction as, the axes of the ECI system we have

$$x''' = x''$$
$$y''' = y'' \tag{7}$$
$$z''' = -z''.$$

It is now necessary to translate the (x''', y''', z''') to the ECI origin (R_x, R_y, R_z). (Figure 3).

$$R_x = x''' + R \cos \theta \cos \Delta\lambda$$
$$R_y = y''' - R \cos \theta \sin \Delta\lambda \tag{8}$$
$$R_z = z''' - R \sin \theta.$$

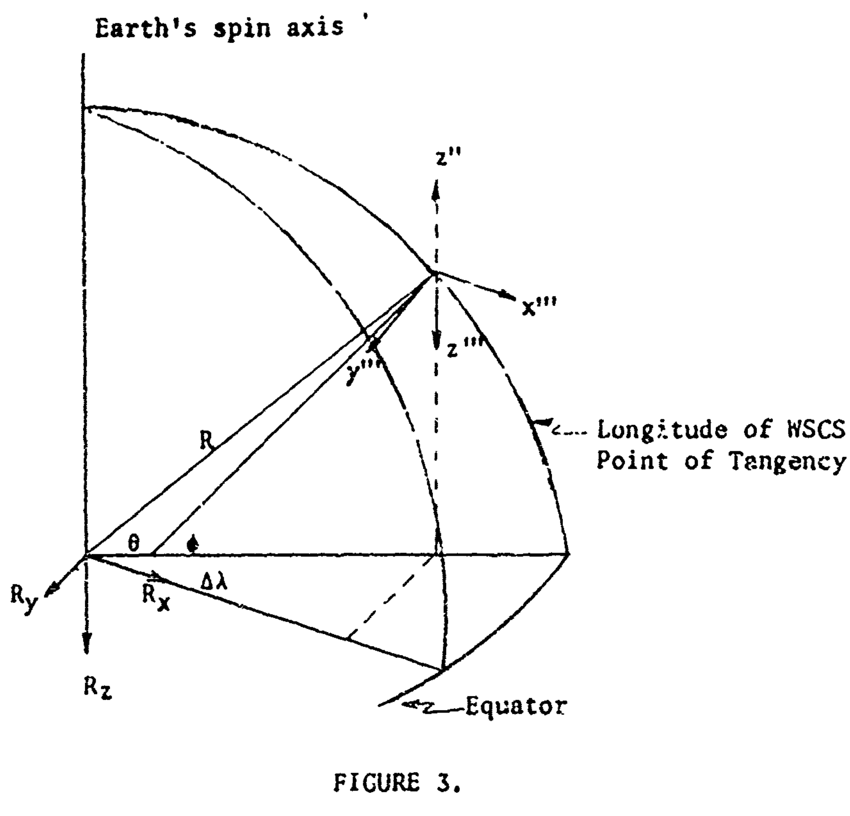

FIGURE 3.

In matrix notation the complete rotation and translation from the left-handed WSCS to the right-handed ECI coordinate system is

$$\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} = \begin{bmatrix} -\cos\Delta\lambda & -\sin\Delta\lambda & 0 \\ \sin\Delta\lambda & -\cos\Delta\lambda & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$+ \begin{bmatrix} R\cos\theta\cos\Delta\lambda \\ -R\cos\theta\sin\Delta\lambda \\ -R\sin\theta \end{bmatrix} \qquad (9)$$

or

$$\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} = \begin{bmatrix} -\cos\Delta\lambda\sin\phi & -\sin\Delta\lambda & \cos\Delta\lambda\cos\phi \\ \sin\Delta\lambda\sin\phi & -\cos\Delta\lambda & -\sin\Delta\lambda\cos\phi \\ -\cos\phi & 0 & -\sin\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$+ \begin{bmatrix} R\cos\theta\cos\Delta\lambda \\ -R\cos\theta\sin\Delta\lambda \\ -R\sin\theta \end{bmatrix}. \qquad (10)$$

Component Velocities

R_x, R_y and R_z may be differentiated with respect to time to determine the component velocities. The component velocities are derived as follows. From equation (10),

$$R_x = -x\cos\Delta\lambda\sin\phi - y\sin\Delta\lambda + z\cos\Delta\lambda\cos\phi + R\cos\theta\cos\Delta\lambda \qquad (11)$$

$$R_y = x\sin\Delta\lambda\sin\phi - y\cos\Delta\lambda - z\sin\Delta\lambda\cos\phi - R\cos\theta\sin\Delta\lambda \qquad (12)$$

$$R_z = -x\cos\phi - z\sin\phi - R\sin\theta. \qquad (13)$$

Differentiating equation (11) with respect to time gives

$$\dot{R}_x = -x(-\sin\Delta\lambda \sin\phi)\frac{d\Delta\lambda}{dt} - (\cos\Delta\lambda\sin\phi)\frac{dx}{dt} - y(\cos\Delta\lambda)\frac{d\Delta\lambda}{dt}$$

$$-(\sin\Delta\lambda)\frac{dy}{dt} + z(-\sin\Delta\lambda\cos\phi)\frac{d\Delta\lambda}{dt} + \frac{dz}{dt}\cos\Delta\lambda\cos\phi$$

$$-R(\sin\Delta\lambda\cos\theta)\left(\frac{d\Delta\lambda}{dt}\right) + \frac{dR}{dt}(\cos\theta\cos\Delta\lambda) \qquad (14)$$

where

$$\frac{d(\Delta\lambda)}{dt} = \omega \qquad \frac{dx}{dt} = \dot{x}$$

$$\frac{dR}{dt} = 0 \qquad \frac{dy}{dt} = \dot{y}$$

$$\frac{d^2(\Delta\lambda)}{dt} = \frac{d\omega}{dt} = \dot{0} \qquad \frac{dz}{dt} = \dot{z},$$

Substituting these values in equation (14) yields

$$\dot{R}_x = \omega x\sin\Delta\lambda\sin\phi - \omega y\cos\Delta\lambda - \omega z\sin\Delta\lambda\cos\phi - \omega R\sin\Delta\lambda\cos\theta$$

$$- \dot{x}\cos\Delta\lambda\sin\phi - \dot{y}\sin\Delta\lambda + \dot{z}\cos\Delta\lambda\cos\phi$$

or

$$\dot{R}_x = \omega[x\sin\Delta\lambda\sin\phi - y\cos\Delta\lambda - z\sin\Delta\lambda\cos\phi - R\cos\theta\sin\Delta\lambda]$$

$$- \dot{x}\cos\Delta\lambda\sin\phi - \dot{y}\sin\Delta\lambda + \dot{z}\cos\Delta\lambda\cos\phi. \qquad (15)$$

From equation (12) we know

$$R_y = x\sin\Delta\lambda\sin\phi - y\cos\Delta\lambda - z\sin\Delta\lambda\cos\phi - R\cos\theta\sin\Delta\lambda. \qquad (12)$$

Substituting this in equation (15) gives

$$\dot{R}_x = \omega R_y - \dot{x}\cos\Delta\lambda\sin\phi - \dot{y}\sin\Delta\lambda + \dot{z}\cos\Delta\lambda\cos\phi. \qquad (16)$$

Differentiating equation (12) in the same manner yields

$$\dot{R}_y = \omega[x\cos\Delta\lambda\sin\phi + y\sin\Delta\lambda - z\cos\Delta\lambda\cos\phi - R\cos\Delta\lambda\cos\theta]$$

$$+ \dot{x}\sin\Delta\lambda\sin\phi - \dot{y}\cos\Delta\lambda - \dot{z}\sin\Delta\lambda\cos\phi. \qquad (17)$$

Substituting equation (11) into equation (17) we have

$$\dot{R}_y = -\omega R_x + \dot{x} \sin \Delta\lambda \sin \phi - \dot{y} \cos \Delta\lambda - \dot{z} \sin \Delta\lambda \cos \phi. \tag{18}$$

Differentiating R_z (equation (13)),

$$\dot{R}_z = -\dot{x} \cos \phi - \dot{z} \sin \phi. \tag{19}$$

Summarizing the velocity components

$$\dot{R}_x = \omega R_y - \dot{x} \cos \Delta\lambda \sin \phi - \dot{y} \sin \Delta\lambda + \dot{z} \cos \Delta\lambda \cos \phi \tag{16}$$

$$\dot{R}_y = -\omega R_x + \dot{x} \sin \Delta\lambda \sin \phi - \dot{y} \cos \Delta\lambda - \dot{z} \sin \Delta\lambda \cos \phi \tag{18}$$

$$\dot{R}_z = - \dot{x} \cos \phi - \dot{z} \sin \phi. \tag{19}$$

In matrix form the velocity components in the ECI coordinate system are:

$$\begin{bmatrix} \dot{R}_x \\ \dot{R}_y \\ \dot{R}_z \end{bmatrix} = \begin{bmatrix} -\cos \Delta\lambda \sin \phi & -\sin \Delta\lambda & \cos \Delta\lambda \cos \phi \\ \sin \Delta\lambda \sin \phi & -\cos \Delta\lambda & -\sin \Delta\lambda \cos \phi \\ -\cos \phi & 0 & -\sin \phi \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \omega \begin{bmatrix} R_y \\ -R_x \\ 0 \end{bmatrix}. \tag{20}$$

Component accelerations

The component accelerations are found by taking the first derivatives of the component velocities and are found in the following manner.

Differentiating equation (15) with respect to time and collecting terms yields

$$\begin{aligned} \ddot{R}_x = {} & \omega^2 x \cos \Delta\lambda \sin \phi + \omega^2 y \sin \Delta\lambda - \omega^2 z \cos \Delta\lambda \cos \phi \\ & -\omega^2 R \cos \Delta\lambda \cos \theta + 2\omega \dot{x} \sin \Delta\lambda \sin \phi - 2\omega \dot{y} \cos \Delta\lambda \\ & - 2\omega \dot{z} \sin \Delta\lambda \cos \phi - \ddot{x} \cos \Delta\lambda \sin \phi - \ddot{y} \sin \Delta\lambda + \ddot{z} \cos \Delta\lambda \cos \phi \end{aligned}$$

or

$$\begin{aligned} \ddot{R}_x = {} & \omega^2 [x \cos \Delta\lambda \sin \phi + y \sin \Delta\lambda - z \cos \Delta\lambda \cos \phi - R \cos \Delta\lambda \cos \theta] \\ & + 2\omega [\dot{x} \sin \Delta\lambda \sin \phi - \dot{y} \cos \Delta\lambda - \dot{z} \sin \Delta\lambda \cos \phi] \\ & - \ddot{x} \cos \Delta\lambda \sin \phi - \ddot{y} \sin \Delta\lambda + \ddot{z} \cos \Delta\lambda \cos \phi. \end{aligned} \tag{21}$$

Using equations (11) and (18) we see that

$$2\omega\dot{R}_y + \omega^2 R_x = \omega^2[x \cos \Delta\lambda \sin \phi + y \sin \Delta\lambda - z \cos \Delta\lambda \cos \phi$$
$$- R \cos \Delta\lambda \cos \theta] + 2\omega[\dot{x} \sin \Delta\lambda \sin \phi - \dot{y} \cos \Delta\lambda$$
$$- \dot{z} \sin \Delta\lambda \cos \phi]. \qquad (22)$$

Making this substitution in equation (21) yields

$$\ddot{R}_x = - \ddot{x} \cos \Delta\lambda \sin \phi - \ddot{y} \sin \Delta\lambda + \ddot{z} \cos \Delta\lambda \cos \phi + 2\omega\dot{R}_y + \omega^2 R_x. \qquad (23)$$

In a like manner $\ddot{R}_y$ and $\ddot{R}_z$ become

$$\ddot{R}_y = \ddot{x} \sin \Delta\lambda \sin \phi - \ddot{y} \cos \Delta\lambda - \ddot{z} \sin \Delta\lambda \cos \phi - 2\omega\dot{R}_x + \omega^2 R_y \qquad (24)$$

$$\ddot{R}_z = - \ddot{x} \cos \phi - \ddot{z} \sin \phi. \qquad (25)$$

In matrix form the acceleration components in the ECI coordinate system become

$$\begin{bmatrix} \ddot{R}_x \\ \ddot{R}_y \\ \ddot{R}_z \end{bmatrix} = \begin{bmatrix} -\cos \Delta\lambda \sin \phi & -\sin \Delta\lambda & \cos \Delta\lambda \cos \phi \\ \sin \Delta\lambda \sin \phi & -\cos \Delta\lambda & -\sin \Delta\lambda \cos \phi \\ -\cos \phi & 0 & -\sin \phi \end{bmatrix} \begin{bmatrix} \ddot{x} \\ y \\ \ddot{z} \end{bmatrix} + \omega \begin{bmatrix} 2\dot{R}_y + \omega R_x \\ -2\dot{R}_x + \omega R_y \\ 0 \end{bmatrix}. \qquad (26)$$

F. WEATHER DATA

F. WEATHER DATA

I Derivative Data and Weather

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DERIVATIVE DATA AND WEATHER

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DERIVATIVE DATA AND WEATHER

Introduction

This program determines ground range, slant range, and height of the missile above mean sea level from trajectory data components. True air speed, indicated air speed, mach number, dynamic pressure, drag acceleration, drag force, and drag coefficient are computed from atmospheric data and trajectory data. Height above sea level as a function of density and of pressure may be computed instead of ground range and slant range.

Air weather information is obtained from a WSMR air weather site and from a WSMR model atmosphere table. If the missile exceeds the altitude of the observed air weather data, then a model upper atmosphere table is used for the higher altitudes. Atmospheric data such as temperature (T), pressure (P), relative humidity (f), wind speed (V_w) and direction from which the wind is coming (θ) are interpolated to the height of the missile.

Telemetry data may be used if available; otherwise vacuum thrust (F_V) and weight of fuel (W_f) will be set to zero for the entire trajectory.

Any available trajectory data in standard DRD format may be used.

The equations derived assume that all quantities are expressed in a consistent system of units (either the English or Metric system for position units, usually the Metric system for meteorological relationships). In computing it is necessary to include conversion factors so that the results will appear in the desired units.

Mathematical Derivations:

Slant Range (R_s) and Ground Range (R_g):

Slant range is defined as the distance from a given position of the missile to the origin. Ground range is the projection of the slant range onto the XY plane.

Let X, Y, Z be the position coordinates of the missile. Then

$$R_s = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$$

$$R_g = (X^2 + Y^2)^{\frac{1}{2}}$$

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Height above Mean Sea Level (H): (See Figure 1)

H = Height above Mean Sea Level (MSL)

H_o = Height of origin above MSL

R_o = Radius of the earth

R_D = Distance from the center of the earth to the missile

To find the height of the missile above MSL it is necessary first to compute the distance from the center of the earth to the missile, and from this, to subtract the radius of the earth.

$$R_D = [X^2 + Y^2 + (R_o + H_o + Z)^2]^{\frac{1}{2}}$$

$$H = R_D - R_o$$

True Air Speed (TAS):

V_x, V_y, V_z = Velocity components of missile at desired altitude

W_x, W_y, W_z = Velocity components of wind at desired altitude

V_w = Wind speed at desired altitude

θ = Direction from which wind is coming at desired altitude

$[M]$ = Rotational matrix (defined in Rotation and Translation Section).

True air speed is defined as the ground speed of the missile corrected for wind velocity. The components of the wind velocity are obtained using the wind speed and wind direction from the air weather data at the desired altitude and the rotational matrix [M]. Since the wind speed from air weather data is given in knots, it must be converted to the (position units/sec) system used for the missile velocity before computation.

$$\begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = [M] \begin{bmatrix} V_w \cos \theta \\ V_w \sin \theta \\ 0 \end{bmatrix}$$

$$TAS = [(V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2]^{\frac{1}{2}}$$

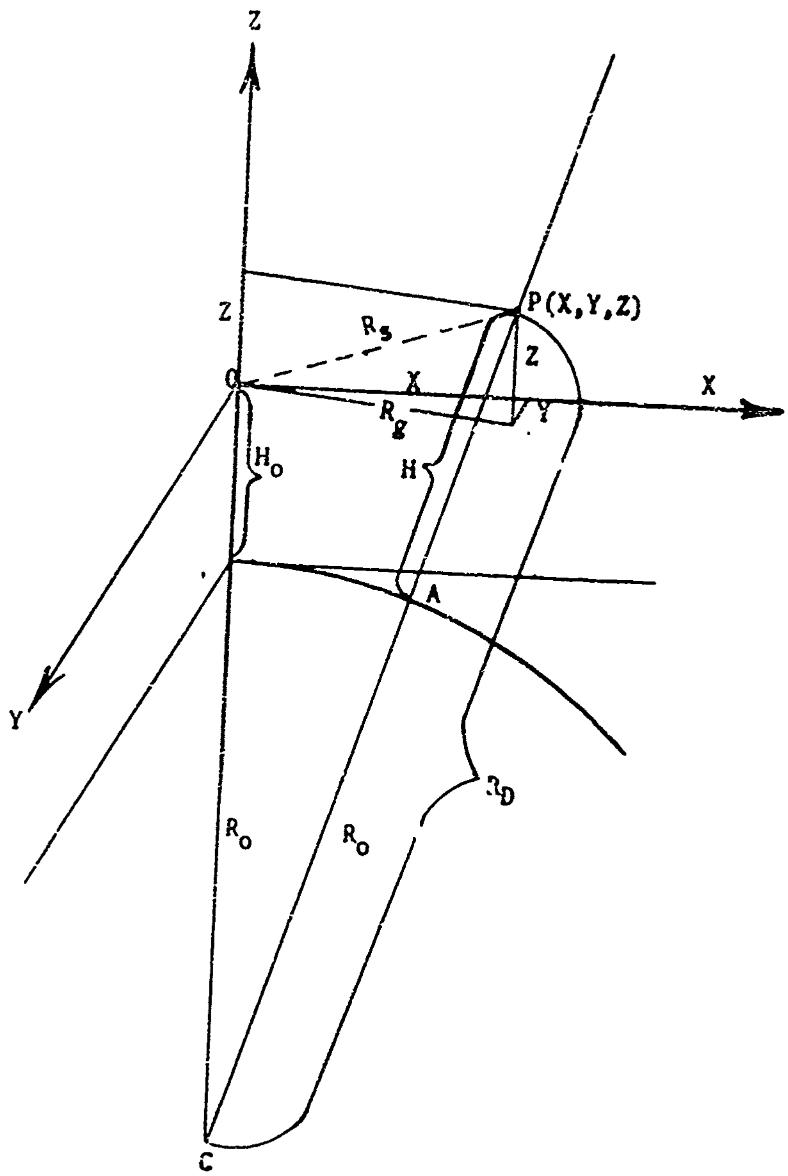

FIG. 1

Tangential Acceleration (A_T):

A_x, A_y, A_z = Acceleration components of missile at desired altitude

Tangential acceleration is the missile acceleration corrected for wind velocity and true air speed.

$$A_T = \frac{1}{TAS}\left[A_x(V_x - W_x) + A_y(V_y - W_y) + A_z(V_z - W_z)\right]$$

Density of Air (ρ):

P = Total air pressure at desired altitude (mb)

P_D = Pressure of dry air at desired altitude (mb)

ρ_D = Density of dry air at desired altitude (gm/m^3)

ρ' = Density of water vapor at desired altitude (gm/m^3)

σ_p = Partial pressure of water vapor

R, R' = gas constants for dry air and water vapor respectively

R^* = universal gas constant

T, T' = Absolute temperature (°K) of dry air and water vapor respectively

m, m' = Molecular weight of dry air and water vapor respectively (gm/mol)

f = Relative humidity (percent)

e_s = Saturation vapor pressure at the temperature of the air in question

C_s = Molar specific heat of a substance

C_i = internal molar specific heat due to rotations and vibrations

λ_o = Latent heat of vaporization

i = A chemical constant of integration

S = Entropy of the substance

Dry air is assumed to obey the perfect gas law, which states the relationship between density, pressure, and temperature. However, the atmosphere is composed of a mixture of dry air and water vapor, each of which occupies the space independently. The density of the mixture is the sum of the density of the dry air and the density of the water vapor.

The ideal gas law gives the density of dry air as

$$\rho_D = \frac{P_D}{RT}$$

or since $P_D = P - e_P$,

$$\rho_D = \frac{P - e_P}{RT}$$

The density of water vapor is given by

$$\rho' = \frac{e_P}{R'\ T'}$$

Since the universal gas constant, R^*, $= mR = m'R'$, then $R' = \frac{mR}{m'}$

and $$\rho' = \frac{m'\ e_P}{m\ RT'}$$

The desired total density, $\rho = \rho_D + \rho'$, is given by

$$\rho = \frac{P - e_P}{RT} + \frac{m'\ e_P}{m\ \ RT'} \quad ,$$

or, since in a mixture of dry air and water vapor, $T = T'$,

$$\rho = \frac{1}{RT}\left[P - e_P + \frac{m'}{m}\, e_P\right]$$

$$= \frac{1}{RT}\left[P + e_P\left(\frac{m'}{m} - 1\right)\right]$$

In this form of the equation all quantities are either known constants (R, m, and m') or weather data observations (P, T), with the exception of the partial pressure of water vapor, e_p. This, however, can be derived from the relative humidity observation (f) and the saturation vapor pressure (e_s). By the definition of relative humidity,

$$f = \frac{e_p}{e_s}$$

Therefore, $e_p = f\, e_s$

The general equation for the vapor pressure of a substance, assuming that the vapor obeys the ideal gas law, is obtained by integration of the Clausius-Clapeyron equation:

$$\ln e_s = -\frac{\lambda_0}{RT} + \frac{S}{2} \ln (T) - \left[\frac{1}{R} \int_0^T \frac{\int_0^T (C_s - C_i)\, dT}{T^2}\, dT \right] + i$$

A special case of this general equation is the Kirchhoff formula, in which the integration is taken between temperature limits sufficiently close for the specific heats to be regarded as constant. This gives the equation

$$\ln e_s = A - \frac{B}{T} - C \ln T.$$

Substituting numerical values for the constants and measuring the temperature in °K, the value of e_s in certibars is found from:

$$\ln e_s = +\left(\frac{-6763.61}{T} - 4.9283 \ln (T) + 51.9274\right)$$

The total density equation can then be written:

$$\rho = \frac{1}{RT}\left[P + f\left(\frac{m'}{m} - 1\right) e^{(\exp)} \right] \text{gm/m}^3$$

where $(\exp) = \left(\frac{-6763.61}{T} - 4.9283 \ln (T) + 51.9274\right)$

Indicated Air Speed (IAS):

Using the values computed for true air speed and total density at the altitude in question, indicated air speed is found from:

$$IAS = TAS \left(\frac{\rho}{\rho_o}\right)^{\frac{1}{2}}$$

where ρ_0 is the standard density of air at sea level.

Mach Number (M):

V_S = Velocity of sound

γ = Ratio of specific heats $\left(\frac{c_p}{c_v}\right)$

c_p = specific heat of dry air at constant pressure

c_v = specific heat of dry air at constant volume

R^* = Universal gas constant

T = Temperature (°K)

m = molecular weight of dry air

Mach number is defined as the ratio of the speed of an object to the speed of sound in the undisturbed medium in which the object is traveling.

The velocity of sound in dry air at the observed temperature T is given by:

$$V_s = \sqrt{\frac{\gamma R^* T}{m}} \text{ cm/sec} = 65.795 \sqrt{T} \text{ ft/sec}$$

Then, mach number is computed from

$$M = \frac{TAS}{V_s} = \frac{TAS}{65.795 \sqrt{T}} = .0151987(TAS)(T)^{-\frac{1}{2}}$$

Dynamic Pressure (q):

Dynamic pressure is the pressure, created by atmospheric pressure and fluid friction, acting on the shell of the missile in flight. Dynamic pressure is computed from Bernoulli's equation of motion for an incompressible fluid, that is,

$$q = \frac{1}{2} \rho \, (TAS)^2$$

Drag (D), Drag Coefficient (C_D), and Drag Acceleration (A_D):

F = atmospheric thrust = $(F_V - f_e \, P)$

F_V = vacuum thrust

f_e = area of the exit nozzle

P = pressure

W = instantaneous mass = $(W_O + W_P)$

W_O = weight of the missile without fuel

W_P = weight of the fuel

g = gravitational acceleration

g_O = gravitational acceleration at sea level, White Sands latitude

θ_P = pitch path angle

s = missile's geometrical cross section area

Drag is a function of thrust (F), instantaneous mass of the missile (W), and drag acceleration (A_D). Drag acceleration depends upon the tangential acceleration (A_T), the pitch path angle (θ_P), and the gravitational acceleration (g) at the latitude (ϕ) and altitude (H).

The gravitational acceleration observed on the earth consists of the actual attraction by the earth diminished by the effect of the centrifugal acceleration caused by the earth's rotation. Since this rotation causes points near the equator to move faster than those at higher latitudes, the centrifugal force decreases as latitude increases. Consequently, the total gravitational acceleration increases with increasing latitude. In addition, the gravitational force at any altitude is inversely proportional to the square of the distance from the center of the earth.

The gravitational acceleration at latitude ϕ and altitude H (in cm) can be expressed by Helmert's equation:

$$g = (980.616 - 2.5928 \cos 2\phi + .0069 \cos^2 2\phi - 3.086 \times 10^{-6} H) \text{ cm/sec}^2$$

The gravitational acceleration at sea level (H=0) and White Sands latitude (ϕ) can then be computed as:

$$g_o = (980.616 - 2.5928 \cos 2\phi + .0069 \cos^2 2\phi) \text{ cm/sec}^2$$

Because of the inverse square relationship between gravitational acceleration and distance from the center of the earth, the following ratio exists:

$$\frac{g}{g_o} = \frac{R_o^2}{R_D^2}$$

or

$$g = g_o \left(\frac{R_o}{R_D}\right)^2$$

Drag acceleration may easily be computed then, from

$$A_D = A_T + g \sin \theta_p$$

or, since $\sin \theta_p = \dfrac{V_z - W_z}{TAS}$,

$$A_D = \frac{1}{TAS}\left[A_x(V_x - W_x) + A_y(V_y - W_y) + A_z(V_z - W_z) + g(V_z - W_z)\right]$$

Drag force (D) is computed from:

$$D = C_3 F - W A_D$$

where C_3 is a multiplier used to correct vacuum thrust for loss of thrust due to jet vanes,

and $F = F_v - f_e P$

$$W = W_o + W_F$$

Finally, the drag coefficient (C_D), a nondimensional quantity, is computed from:

$$C_D = \frac{D}{qs}$$

Height above MSL as a function of Pressure (H_P)

a = lapse rate = rate of change of temperature with altitude

H = height above MSL or geopotential height

Z = altitude

G = dimensional constant

g = gravitational acceleration

T_0 = Standard temperature at sea level

P_0 = Standard pressure at sea level

T^* = Temperature defining the Tropopause and constant temperature of the stratosphere

P^* = Pressure at tropopause

H^* = Height of tropopause

The earth's atmosphere consists of the troposphere, tropopause and stratosphere. The troposphere is that part of the atmosphere in which temperature generally decreases with altitude, clouds form, and convection occurs. It occupies the space above the earth's surface up to the tropopause.

The tropopause is defined as the discontinuity surface separating the stratosphere from the troposphere. It varies in height from about 55,000 feet at the equator to 25,000 feet at the poles.

The stratosphere is that portion of the earth's atmosphere above the tropopause. This air is free from all weather phenomena, practically without moisture, and in general, an isothermal structure.

The method of computing height above MSL as a function of pressure differs for the troposphere and stratosphere.

(A) In the Troposphere

The relationship between height and pressure in the troposphere is dependent upon three fundamental relationships: the ideal gas law, the lapse rate or change of temperature with altitude, and the hydrostatic equation relating pressure, density, gravitational force and altitude.

Because of the changing gravitational force with altitude, the hydrostatic equation can be stated in terms of geopotential height rather than simply geometric height. Geopotential height is the height of a given point in the atmosphere in units proportional to the potential energy of unit mass at this height, relative to sea level. This relationship between geopotential and geometric height is given by $G\,dH = g\,dZ$.

From this, the hydrostatic equation

$$dP = -\rho\, g\, dZ$$

can be written

$$dP = -\rho\, G\, dH. \tag{1}$$

The lapse rate, or change of temperature with change of height in the troposphere, is defined as $a = -\frac{dT}{dH}$. The temperature at the height H is given by $T = T_0 - aH$.

Finally, the ideal gas law, $\rho = \frac{P}{RT}$, can be written as

$$P = \rho RT = \rho R(T_0 - aH). \tag{2}$$

From equations (1) and (2) we can form the ratio

$$\frac{dP}{P} = \frac{-\rho G\, dH}{\rho R\,(T_0 - aH)} = \frac{-G\,(dH)}{R\,(T_0 - aH)}$$

or, since $dH = \frac{-dT}{a}$,

$$\frac{dP}{P} = \frac{G\,(dT)}{aR\,(T_0 - aH)} = \frac{G}{aR}\,\frac{d\,(T_0 - aH)}{(T_0 - aH)} \tag{3}$$

Equation (3) integrated between the limits of 0 and H becomes

$$\int_0^H \frac{dP}{P} = \int_0^H \frac{G}{aR} \frac{d\,(T_O - aH)}{(T_O - aH)}$$

or

$$\ln\left(\frac{P}{P_O}\right) = \frac{G}{aR} \ln\left(\frac{T_O - aH}{T_O}\right) \tag{4}$$

If $\frac{G}{aR}$ is set equal to n, equation (4) yields

$$\left(\frac{P}{P_O}\right) = \left(\frac{T_O - aH}{T_O}\right)^n \tag{5}$$

Height as a function of pressure, H_P, is found by solving equation (5) for $H = H_P$.

$$H_P = \frac{T_O}{a}\left[1 - \left(\frac{P}{P_O}\right)^{1/n}\right]. \tag{6}$$

(B) In the Stratosphere

The tropopause is by definition at the height H* such that the temperature $T^* = (T_O - aH)$ is a constant, 216.66°K. The temperature in the stratosphere is assumed to remain constant at $T = T^*$.

Thus, in the stratosphere, equation (3) becomes

$$-\frac{dP}{P} = \frac{G}{RT^*} (dH) \tag{7}$$

Integrating this between the limits H* and H yields

$$\ln\left(\frac{P^*}{P}\right) = \frac{G}{RT^*} (H - H^*) \tag{8}$$

Using common logarithms, equation (8) becomes

$$\log_{10}\left(\frac{P^*}{P}\right) = \left(\frac{G}{RT^*} \log_{10} e\right) (H - H^*)\ . \tag{9}$$

Letting $\dfrac{G \log_{10} e}{RT^*} = B$,

$$\log_{10}\left(\frac{P^*}{P}\right) = B\,(H - H^*) \tag{10}$$

The known value of H^* can be substituted in equation (5) to find the pressure at the tropopause:

$$\frac{P^*}{P_o} = \left(\frac{T_o - aH^*}{T_o}\right)^n$$

or

$$\log_{10}\left(\frac{P^*}{P_o}\right) = n \log_{10}\left(\frac{T_o - aH^*}{T_o}\right) = n \log_{10}\left(\frac{T^*}{T_o}\right) \tag{11}$$

Equations (10) and (11) can be combined and solved for $H = H_p$.

$$\log_{10}\left(\frac{P^*}{P_o}\right) - \log_{10}\left(\frac{P^*}{P}\right) = n \log_{10}\left(\frac{T^*}{T_o}\right) - B(H - H^*)$$

$$\log_{10}\left(\frac{P}{P_o}\right) = n \log_{10}\left(\frac{T^*}{T_o}\right) - BH + BH^*$$

$$H = H^* + \frac{n}{B}\log_{10}\left(\frac{T^*}{T_o}\right) - \frac{1}{B}\log_{10}\left(\frac{P}{P_o}\right)$$

Thus, in the stratosphere, height as a function of pressure is found from

$$H_p = H^* + \frac{1}{B}\left[n \log_{10}\left(\frac{T^*}{T_o}\right) - \log_{10}\left(\frac{P}{P_o}\right)\right]$$

Height above MSL as a function of density (H_ρ):

(A) In the Troposphere:

Differentiation of the ideal gas law, $\rho = \frac{P}{RT}$, yields

$$d\rho = \frac{dP}{RT} - \frac{PdT}{RT^2} = \frac{1}{RT}\left(dP - \frac{PdT}{T}\right)$$

and the ratio

$$\frac{d\rho}{\rho} = \left(\frac{dP}{P} - \frac{dT}{T}\right)$$

In the previous section (height as a function of pressure in the troposphere) equation (3) gave the expression for

$$\frac{dP}{P} = \frac{G}{aR}\,\frac{dT}{T}\ .$$

Substituting this, we find

$$\frac{d\rho}{\rho} = \frac{G}{aR}\,\frac{dT}{T} - \frac{dT}{T}$$

$$= \left(\frac{G}{aR} - 1\right)\frac{dT}{T} = (n-1)\,\frac{dT}{T}\ .$$

Since in the troposphere $T = T_o - aH$

$$\frac{d\rho}{\rho} = (n-1)\,\frac{d\,(T_o - aH)}{(T_o - aH)}$$

Integrating over the limits 0 to H

$$\ln\left(\frac{\rho}{\rho_o}\right) = (n-1)\,\ln\left(\frac{T_o - aH}{T_o}\right)$$

or

$$\left(\frac{\rho}{\rho_o}\right) = \left(\frac{T_o - aH}{T_o}\right)^{n-1}$$

Solving for $H = H_\rho$ yields:

$$H_\rho = \frac{T_0}{a}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{(n-1)^{-1}}\right]$$

(B) In the Stratosphere

Since the stratosphere is assumed to have a constant temperature T^*, the ideal gas law becomes

$$\rho = \frac{P}{RT^*}$$

The standard density of air at sea level (zero altitude), ρ_0, may be expressed as

$$\rho_0 = \frac{P_0}{RT_0}$$

Taking the common logarithm of the ratio $\frac{\rho}{\rho_0}$ yields the relationship:

$$\frac{\rho}{\rho_0} = \frac{T_0 P}{T^* P_0}$$

$$\log_{10}\left(\frac{\rho}{\rho_0}\right) = \log_{10}\left(\frac{T_0}{T^*}\right) + \log_{10}\left(\frac{P}{P_0}\right)$$

By substituting in this equation the expression found previously (H_p in the stratosphere) for $\log_{10}\left(\frac{P}{P_0}\right) = \left[n \log_{10}\left(\frac{T^*}{T_0}\right) - B(H - H^*)\right]$

we find:

$$\log_{10}\left(\frac{\rho}{\rho_0}\right) = \log_{10}\left(\frac{T_0}{T^*}\right) + n \log_{10}\left(\frac{T^*}{T_0}\right) - B(H - H^*)$$

$$\log_{10}\left(\frac{\rho}{\rho_0}\right) = (n-1) \log_{10}\left(\frac{T^*}{T_0}\right) - B(H - H^*)$$

The term $\left[(n-1)\ \log_{10}\left(\frac{T^*}{T_o}\right)\right]$ is a constant, equal to

$$\log_{10}\left(\frac{\rho^*}{\rho_o}\right) = -0.527139.$$

Substituting this and solving for $H = H_\rho$ yields the final equation for height as a function of density in the stratosphere:

$$\log_{10}\left(\frac{\rho}{\rho_o}\right) = \log_{10}\left(\frac{\rho^*}{\rho_o}\right) - BH + BH^*$$

$$H_\rho = H^* + \frac{1}{B}\log_{10}\left(\frac{\rho^*}{\rho_o}\right) - \frac{1}{B}\log_{10}\left(\frac{\rho}{\rho_o}\right)$$

TABLE OF CONSTANTS

Radius of the earth	R_0	20,897,038.00 ft
Dry air gas constant	R	2.8704×10^6 $cm^2/sec^2(°K)$
Molecular weight of dry air	m	28.966 gm/mol
Molecular weight of water vapor	m'	18.016 gm/mol
Ratio of specific heats	γ	1.40112 (dimensionless)
Gravitational acceleration at sea level, White Sands latitude	g_0	32.138 ft/sec^2 = 979.569 cm/sec^2
Lapse rate	a	.0065 °C/m
Dimensional constant	G	32.17405 ft/sec^2
Standard temperature at sea level	T_0	288.16°K
Standard pressure at sea level	P_0	1013.25 mb
Standard density at sea level	ρ_0	1225.00 gm/m^3
Temperature at Tropopause	T^*	216.66°K
Pressure at Tropopause	P^*	226.32 mb
Density at Tropopause	ρ^*	363.92 gm/m^3
Height of Tropopause at White Sands latitude	H^*	11,000 m = 36,089.227 ft
Constant = $\dfrac{G}{aR}$	n	5.2561 (dimensionless)
Constant = $\dfrac{G \log_{10} e}{RT^*}$	B	0.2087367×10^{-4}/ft

Computational Steps:

From position data at the j^{th} time (X_j, Y_j, Z_j):

Ground range $R_{g_j} = (X_j^2 + Y_j^2)^{1/2}$ pos. units (1)

Slant range $R_{s_j} = (X_j^2 + Y_j^2 + Z_j^2)^{1/2}$ pos. units (2)

Height above MSL

$$R_{D_j} = \left[X_j^2 + Y_j^2 + \left(\frac{R_0 + H_0}{C_1} + Z_j\right)^2\right]^{1/2} \quad \text{pos. units} \quad (3)$$

$$H_j = R_{D_j} - \left(\frac{R_0}{C_1}\right) \quad \text{pos. units} \quad (4)$$

where C_1 is a multiplier to convert position data to feet.

Interpolate the atmospheric data (Temperature, pressure, relative humidity, wind velocity and wind direction) to H_j.

Interpolate telemetry data (vacuum thrust and weight of fuel) to the time of the position data.

The linear interpolation program used is as described in the "Mathematical Miscellaneous Section" of this report.

Gravitational acceleration

$$g_j = 32.138 \left(\frac{R_0}{C_1 \, R_{D_j}}\right)^2 \text{ feet/sec}^2 \quad (5)$$

Density

$$\rho_j = 348.38 \left[\frac{P_j - 0.0378 \, f_j \, e^{(\exp_j)}}{T_j + 273.16}\right] \text{grams/m}^3$$

$$\text{where } (\exp_j) = \left(\frac{-6763.61}{T_j + 273.16} - 4.9283 \ln(T_j + 273.16) + 51.9274\right) \quad (6)$$

Wind Velocity components

$$\begin{bmatrix} W_{X_j} \\ \\ W_{Y_j} \\ \\ W_{Z_j} \end{bmatrix} = [M] \begin{bmatrix} \dfrac{-1.6889639}{C_1} V_j \cos \theta_j \\ \\ \dfrac{-1.6889639}{C_1} V_j \sin \theta_j \\ \\ 0 \end{bmatrix} \quad \text{pos. units/sec.} \quad (7)$$

where -1.6889639 is the conversion factor from knots to feet per second.

True Air speed

$$TAS_j = \left[(V_{Xj} - W_{Xj})^2 + (V_{Yj} - W_{Yj})^2 + (V_{Zj} - W_{Zj})^2 \right]^{\frac{1}{2}} \quad \text{pos. units/sec.} \quad (8)$$

Indicated Air speed

$$IAS_j = (TAS)_j \left(\frac{\rho j}{1225.00} \right)^{\frac{1}{2}} \quad \text{pos. units/sec.} \quad (9)$$

Mach Number

$$M_j = (0.0151987) \; (C_1) \; (TAS)_j \; [T_j^* + 273.16]^{-\frac{1}{2}} \quad \text{dimensionless} \quad (10)$$

Dynamic Pressure

$$q_j = (9.701661) \; [C_1 \; (TAS)_j]^2 \; (\rho_j) \; 10^{-7} \; \text{lbs/ft}^2 \quad (11)$$

Tangential Acceleration

$$A_{T_j} = \frac{A_{Xj}(V_{Xj} - W_{Xj}) + A_{Yj}(V_{Yj} - W_{Yj}) + A_{Zj}(V_{Zj} - W_{Zj})}{(TAS)_j} \quad (12)$$

pos. units/sec^2 or G's

Drag Acceleration

$$A_{D_j} = A_{T_j} + \frac{C_4\,(V_{Z_j} - W_{Z_j})}{(TAS)_j} \quad \text{pos units/sec}^2 \text{ or G's} \qquad (13)$$

If A_{T_j} is in G's, $C_4 = 1.0$

If A_{T_j} is in (units/sec^2), $C_4 = \frac{g_j}{C_1}$

Atmospheric thrust

$F_j = 0$, if no telemetry data available (14a)

$$F_j = C_3\,F_V - (2.088576)\,f_e\,P_j \qquad \text{(lbs)} \qquad (14b)$$

where C_3 is a multiplier to correct vacuum thrust for loss of thrust due to jet vanes, and 2.088576 is a conversion factor from millibars to lbs/ft^2.

Drag $$D_j = F_j - (W_o + W_{F_j})\,\frac{A_{D_j}}{C_5} \qquad \text{(lbs)} \qquad (15)$$

If A_{T_j} is in G's, $C_5 = \frac{32.174}{g_j}$

If A_{T_j} is in (units/sec^2), $C_5 = \frac{32.174}{C_1}$

Drag coefficient

$$C_{D_j} = \frac{D_j}{q_j\,S} \qquad \text{dimensionless} \qquad (16)$$

Height above MSL as a function of Pressure

In Troposphere:

$$H_{p_j} = \frac{T_o}{a}\left[1 - \left(\frac{P_j}{P_o}\right)^{1/n}\right]$$

or

$$H_{p_j} = 145{,}446.67\left[1 - e^{0.190255132\,\ln\left(\frac{P_j}{P_o}\right)}\right] \qquad \text{pos. units (17a)}$$

In Stratosphere

$$H_{p_j} = H^* + \frac{1}{B}\left[n\,\log_{10}\left(\frac{T^*}{T_o}\right)\right] - \frac{1}{B}\log_{10}\left(\frac{P_j}{P_o}\right)$$

or

$$H_{p_j} = 4{,}901.8987 - 20{,}805.8517\,\ln\left(\frac{P_j}{P_o}\right) \qquad \text{pos. units (17b)}$$

Height above MSL as a function of Density

In Troposphere:

$$H_{\rho_j} = \frac{T_o}{a}\left[1 - \left(\frac{\rho_j}{\rho_o}\right)^{1/(n-1)}\right]$$

or

$$H_{\rho_j} = 145{,}446.67\left[1 - e^{0.234956885\,\ln\left(\frac{\rho_j}{\rho_o}\right)}\right] \qquad \text{pos. units (18a)}$$

In Stratosphere:

$$H_{\rho_j} = H^* + \frac{1}{B}\log_{10}\left(\frac{\rho^*}{\rho_o}\right) - \frac{1}{B}\log_{10}\left(\frac{\rho}{\rho_o}\right)$$

or

$$H_{\rho_j} = 10{,}834.9234 - 20{,}805.8517\,\ln\left(\frac{\rho_j}{\rho_o}\right) \qquad \text{pos. units (18b)}$$

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I Data Editing Routine

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DATA EDITING ROUTINE

Introduction

The purpose of this routine is to find and eliminate gross errors (blunders) from a set of equally-spaced data points and, where sufficient "good" points are available about a rejected point, to compute a "replacement" point for that data sample.

The test is based on the assumption that there is no significant change in the variance of the data over a group of N consecutive points (T_i, X_i). Two sets of fourth differences squared are computed, ΔC_i using consecutive points and ΔE_i using every other point. The variance of each set is computed and the individual differences tested for significant variations, using the Snedecor F-test. Differences which are greater than the acceptable limit are set equal to zero. The data points themselves are then accepted as "good" or rejected as "bad" depending upon the number and positions of zeros in the difference sets. A rejected point is replaced by solving for the midpoint of a 2nd degree curve fitted to the data about the rejected point.

Theory

The sets of fourth differences squared are computed from the following:

$$\Delta C_i = (X_i - 4X_{i-1} + 6X_{i-2} - 4X_{i-3} + X_{i-4})^2$$

$$\Delta E_i = (X_i - 4X_{i-2} + 6X_{i-4} - 4X_{i-6} + X_{i-8})^2$$

If the data includes a blunder error (E) at point i, it will appear as $(\text{NOISE} + E)^2$ in the ΔC's at i and i+4, as approximately $16E^2$ at i+1 and i+3, and as approximately $36E^2$ at i+2. In the ΔE's the error will appear as $(\text{NOISE} + E)^2$ at i and i+8, as approximately $16E^2$ at i+2 and i+6, and as approximately $36E^2$ at i+4. Therefore if the ΔC's at points i+1, i+2, and i+3 and the ΔE's at i+2, i+4 and i+6 are all greater than the critical level determined from the variance of the differences and the tables of Snedecor F-test values, it is probable that the data point X_i contains a gross error.

The variance of the ΔC's (σ_C^2) and the variance of the ΔE's (σ_E^2) are computed from:

$$\sigma_C^2 = \frac{\sum_{i=5}^{N} \Delta C_i}{N-4} \quad \text{and} \quad \sigma_E^2 = \frac{\sum_{i=9}^{N} \Delta E_i}{N-8}$$

where (N-4) and (N-8) are the respective degrees of freedom.

A. Testing ΔC's and ΔE's

The individual ΔC's to be rejected are found using the Snedecor F-test. The value of F is selected whose degrees of freedom for the numerator equals one, and whose degrees of freedom for the denominator equals (N-4), the number of ΔC's. Then any $\Delta C_i \geq \sigma_C^2 \, (F_{1,DF})$ should be rejected. F is computed using a 95% confidence level.

If one or more ΔC's are rejected they are replaced with zeroes, and the degrees of freedom decreased by one for each ΔC rejected. Then a new variance, σ_C^2, is computed, a new value of F selected, and the test repeated until no additional ΔC's are rejected.

The ΔE's are tested and rejected in a similar manner.

If the variance of the data (without the blunder error points) is desired it is easily computed from the variate difference procedure since

$$\sigma_X^2 = \frac{(K!)^2}{(2K)!} \; \frac{\Sigma(\Delta_i^K)^2}{(N-K)} = \frac{(K!)^2}{(2K)!} \; \sigma_C^2$$

where K is the order of difference taken.

Using fourth differences this becomes

$$\sigma_X^2 = \frac{(4!)^2}{(2x4)!} \; \sigma_C^2 = \frac{\sigma_C^2}{70}$$

B. Testing the Data Points X_i

In testing the data points from i=1 to N it is necessary to assume that the ΔC's exist and have not been rejected for i=1 thru i=4 and i=N+1 thru i=N+4, and also that the ΔE's exist and have not been rejected for the points i=3 thru i=8 and i=N+1 thru i=N+6. This forces the test to accept the first three and the last three points as being good data points.

If an error exists at point i we could expect the ΔC's at points i, i+1, i+2, i+3, and i+4 to be rejected, and also the ΔE's at i, i+2, i+4, i+6, and i+8 to be rejected. If the error is small the ΔC's at i and i+4 and the ΔE's at i and i+8 may not be rejected.

Therefore if all 6 values (ΔC_{i+1}, ΔC_{i+2}, ΔC_{i+3}, ΔE_{i+2}, ΔE_{i+4}, ΔE_{i+6}) have been rejected, there is probably an error at point i. If none of these, or only some, have been rejected, then the point X_i is accepted as a good data point.

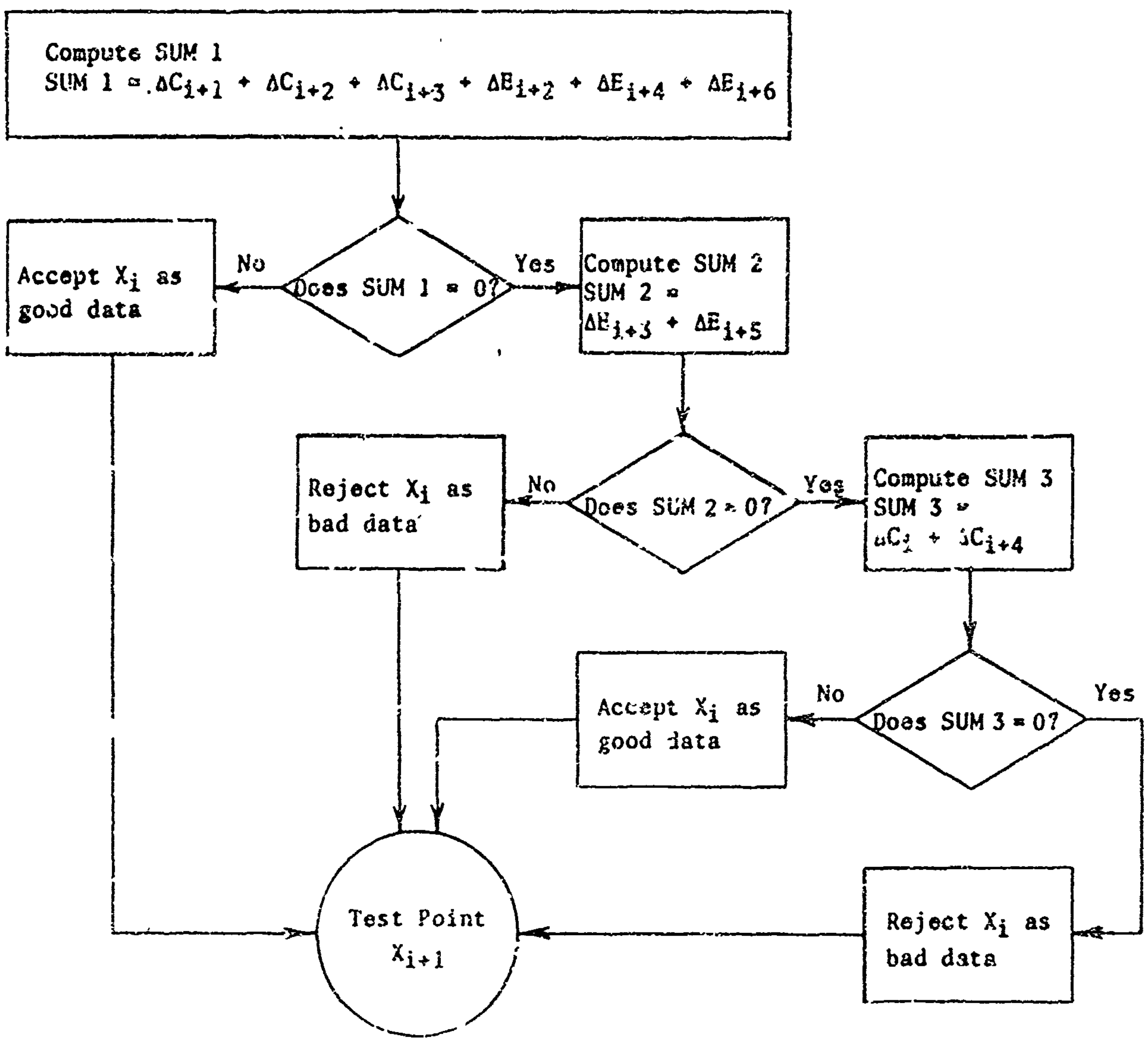
Since two consecutive errors at points i-2 and i-1 or at points i+1 and i+2 could also cause these six Δ's to be rejected, it is necessary to test further.

If the point i+1 or i-1 were in error both ΔE_{i+3} and ΔE_{i+5} would be rejected. But if either of these has not been rejected, we assume that the error detected is at data point i.

If both ΔE's have been rejected, we test ΔC_i and ΔC_{i+4}. If these have both been rejected, data point i is assumed to be in error. If either or both has been accepted then data point i is accepted as a good point, and we go on to testing for errors at point i+1.

This logic is more clearly explained by the following diagram:

TESTING POINT i

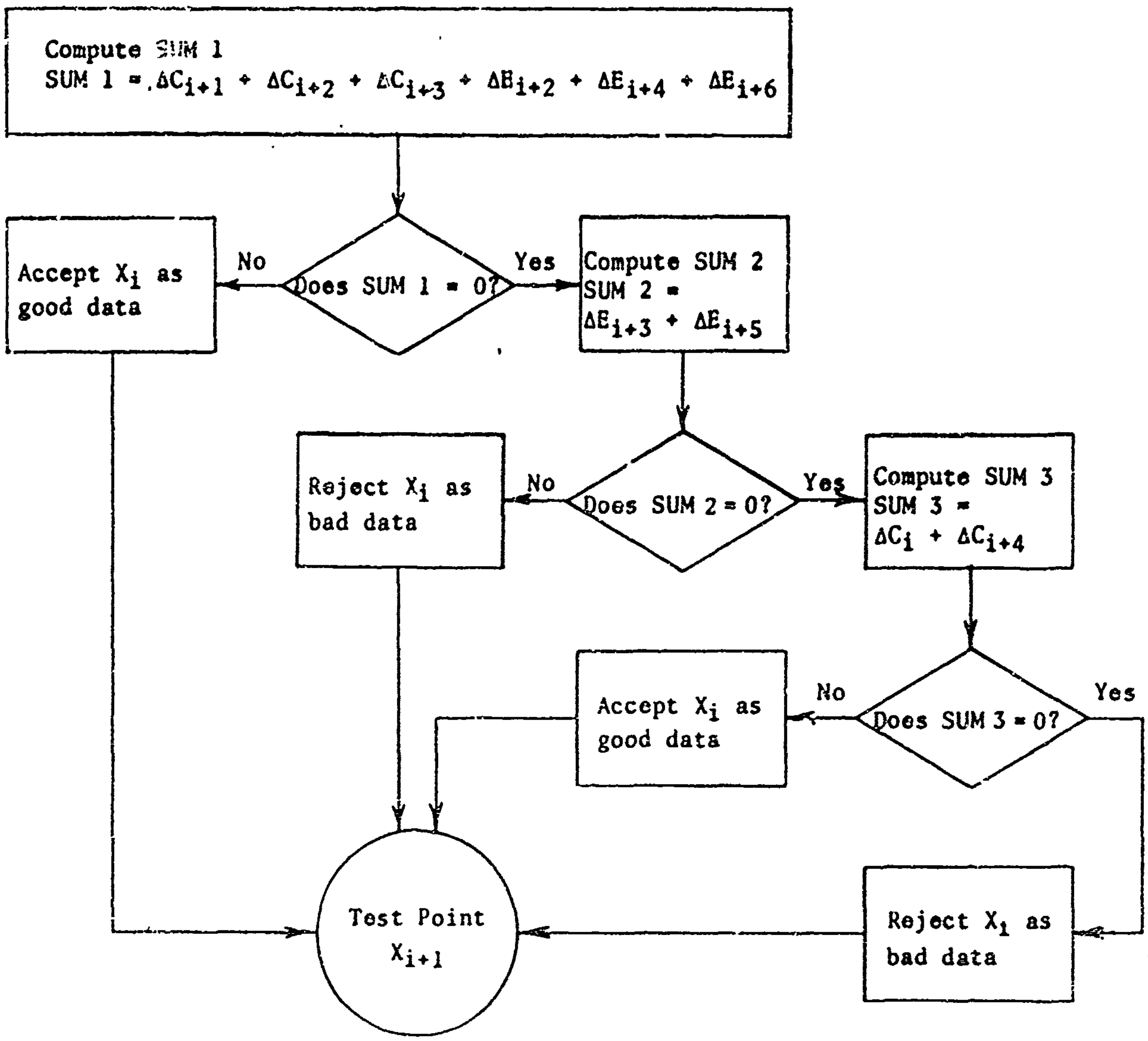


If the point i+1 or i-1 were in error both ΔE_{i+3} and ΔE_{i+5} would be rejected. But if either of these has not been rejected, we assume that the error detected is at data point i.

If both ΔE's have been rejected, we test ΔC_i and ΔC_{i+4}. If these have both been rejected, data point i is assumed to be in error. If either or both has been accepted then data point i is accepted as a good point, and we go on to testing for errors at point i+1.

This logic is more clearly explained by the following diagram:

TESTING POINT i



The above tests will work in most cases. However, as will happen in any editing procedure, situations can arise where good points will be indicated as being in error and points with errors only slightly larger than the noise may go undetected. The test will also work through points of discontinuity since the event itself will usually produce only one or two Δ's that will be rejected. To avoid deleting good points a further check is made. From the surrounding data a replacement point, X_{s_i}, is computed. If $(X_{s_i} - X_i)^2 < (F_{1,DF})\sigma_X^2$ then X_i will be retained. F is computed using a 99% confidence level.

C. Replacement of Bad Points

The point in error is replaced by solving for the midpoint of a 2nd degree curve fitted to the data about the point in error. A maximum of 3 points before and 3 points after the point to be replaced are used in a weighted least squares procedure.

The data points are assigned weights. If the data point at i is accepted as good the weight is set equal to one. If the data point at i is in error the weight is set equal to zero. Any data point whose weight is equal to zero is replaced if there are sufficient data before and after the point.

Let the data point to be replaced be at point i. Then there are sufficient data to replace the point if

$$\sum_{t=i-3}^{t=i+3} W_t \geq 3 \text{ and } \sum_{t=i-3}^{t=i-1} W_t \geq 1 \text{ and } \sum_{t=i+1}^{t=i+3} W_t \geq 1$$

The replacement point X_{s_i} is computed from the following equations. All summations are from t=i-3 to t=i+3.

$$X_{s_i} = \frac{A(S6) - B(S7) + C(S8)}{A(S1) - B(S2) + C(S3)}$$

where

$$A = (S3)(S5) - (S4)(S4)$$

$$B = (S2)(S5) - (S3)(S4)$$

$$C = (S2)(S4) - (S3)(S3)$$

and

$$S1 = \Sigma W_t$$

$$S2 = \Sigma W_t\ (t-i)$$

$$S3 = \Sigma W_t\ (t-i)^2$$

$$S4 = \Sigma W_t\ (t-i)^3$$

$$S5 = \Sigma W_t\ (t-i)^4$$

$S6 = \Sigma X_t \, W_t$

$S7 = \Sigma X_t \, W_t \, (t-i)$

$S8 = \Sigma X_t \, W_t \, (t-i)^2$

Procedure

In this application there are usually M consecutive sets of N data points. Although the editing routine assumes automatically that the 3 points at the beginning and end of each set are good data points, by overlapping the testing of adjoining sets the number of data points accepted without testing is minimized.

In editing the first set we use the first (N+9) points and compute ΔC's and ΔE's thru $i=N+9$. This allows us to compute the W_i's thru $i=N+3$ and replace if necessary data points thru $i=N$.

To edit each succeeding set we must have available from the previous set the last 12 data points from $i=N-2$ to $i=N+9$, the last 6 W_i from $i=N-2$ to $i=N+3$, and the last 9 ΔC's and ΔE's from $i=N+1$ to $i=N+9$.

1. Let N=50

2. Compute ΔC's and ΔE's.

   $\Delta C_i = (X_i - 4X_{i-1} + 6X_{i-2} - 4X_{i-3} + X_{i-4})^2$, i=5 to N+9

   $\Delta E_i = (X_i - 4X_{i-2} + 6X_{i-4} - 4X_{i-6} + X_{i-8})^2$, i=9 to N+9

   Set ΔC_4, ΔE_6, ΔE_7, and ΔE_8 = .0000001

3. If any computed $\Delta=0$, set $\Delta=.0000001$

4. Compute degrees of freedom:

   $DF_C = (N+9)-4 = N+5$

   $DF_E = (N+9)-8 = N+1$

5. Compute variances

$$\sigma_C^2 = \frac{\sum_{i=5}^{N+9} \Delta C_i}{DF_C}$$

$$\sigma_E^2 = \frac{\sum_{i=9}^{N+9} \Delta E_i}{DF_E}$$

6. Compute value of F for F test (95%)

$$F_C = 3.799 + \frac{11.73}{DF_C}$$

$$F_E = 3.799 + \frac{11.73}{DF_E}$$

7. Compute critical values

$CV_C = \sigma_C^2 \ (F_C)$

$CV_E = \sigma_E^2 \ (F_E)$

8. TEST: If $\Delta C_i \geq CV_C$, set $\Delta C_i = 0$, i=5 to N+9.

If $\Delta E_i \geq CV_E$, set $\Delta E_i = 0$, i=9 to N+9.

9. If any ΔC's were set equal to zero in test 8, decrease the degrees of freedom by the number of Δ's set equal to zero, recompute σ_C^2, F_C, CV_C, and repeat test 8.

If any ΔE's were set equal to zero in test 8, decrease the degrees of freedom by the number of Δ's set equal to zero, recompute σ_E^2, F_E, CV_E, and repeat test 8.

Repeat step 9 until no more Δ's are set equal to zero.

10. Compute variance of data and its critical value

$$\sigma_X^2 = \frac{\sigma_C^2}{70} \qquad CV_X = \frac{\sigma_C^2}{70}\left(6.40 + \frac{36.4}{DF_C}\right)$$

11. Test data points X_i, i=4 to N+3 (see diagram)

SUM 1(i) = $\Delta C_{i+1} + \Delta C_{i+2} + \Delta C_{i+3} + \Delta E_{i+2} + \Delta E_{i+4} + \Delta E_{i+6}$

If SUM 1(i) $\neq 0$, set $W_i = 1$ and go to (i+1)

If SUM 1(i) = 0, compute:

SUM 2(i) = $\Delta E_{i+3} + \Delta E_{i+5}$

If SUM 2(i) $\neq$ 0, set W_i = 0 and go to (i+1)

If SUM 2(i) = 0, compute:

SUM 3(i) = $\Delta C_i + \Delta C_{i+4}$

If SUM 3(i) $\neq$ 0, set W_i = 1

If SUM 3(i) = 0, set W_i = 0

On first set of N points set W_i = 1 for i=1, 2, 3.

Go to (i+1).

12. Replace bad points X_i, i=4 to N

If W_i = 1, $X_i = X_i$

If W_i = 0, compute new X_i as follows:

$$S1A = (W_{i-3} + W_{i-2} + W_{i-1})$$

$$S1B = (W_{i+1} + W_{i+2} + W_{i+3})$$

TEST: Is ($S1A \geq 1$) and ($S1B \geq 1$) and ($S1A + S1B \geq 3$)?

If NO, then X_i can not be replaced.

If YES, let S1 = (S1A + S1B) and continue:

$$S2 = \sum_{t=i-3}^{t=i+3} W_t\ (t-i)$$

$$S3 = \sum_{t=i-3}^{t=i+3} W_t\ (t-i)^2$$

$$S4 = \sum_{t=i-3}^{t=i+3} W_t\ (t-i)^3$$

$$S5 = \sum_{t=i-3}^{t=i+3} W_t\ (t-i)^4$$

$$S6 = \sum_{t=i-3}^{t=i+3} X_t W_t$$

$$S7 = \sum_{t=i-3}^{t=i+3} X_t W_t (t-i)$$

$$S8 = \sum_{t=i-3}^{t=i+3} X_t W_t (t-i)^2$$

$A = (S3)(S5) - (S4)(S4)$

$B = (S2)(S5) - (S3)(S4)$

$C = (S2)(S4) - (S3)(S3)$

Then $X_{s_i} = \dfrac{A(S6) - B(S7) + C(S8)}{A(S1) - B(S2) + C(S3)}$

If $(X_{s_i} - X_i)^2 \geq CV_X$, Replace X_i with X_{s_i}.

13. To edit each succeeding set we must have available from the previous set:

the last 12 X_i's from $i=N-2$ to $i=N+9$

the last 6 W_i's from $i=N-2$ to $i=N+3$

the last 9 ΔC's and ΔE's from $i=N+1$ to $i=N+9$

14. If the last set of points does not have 50 points, enough points from the $(M-1)^{th}$ set are included to make the proper number.

APPENDIX I. F-test for Data Editing

F-tests for the data editing routine are approximate F's found by linear interpolation for values of F in the interval of 10 to 40 degrees of freedom for both a 95% confidence level and a 99% confidence level.

For 95%:

$$F_{10} = 4.96$$

$$F_{40} = 4.08$$

$$F_{DF} = 4.96 + (4.08 - 4.96)\,\frac{\left(\frac{1}{DF} - \frac{1}{10}\right)}{\left(\frac{1}{40} - \frac{1}{10}\right)}$$

$$= 4.96 + \frac{(-.88)}{\left(-\frac{3}{40}\right)}\left(\frac{1}{DF} - \frac{1}{10}\right)$$

$$= 4.96 + \frac{35.20}{3}\left(\frac{1}{DF} - \frac{1}{10}\right)$$

$$= 4.96 + \frac{11.73}{DF} - 1.17$$

$$= 3.79 + \frac{11.73}{DF}$$

For 99%:

$$F_{10} = 10.04$$

$$F_{40} = 7.31$$

$$F_{DF} = 10.04 + (7.31 - 10.04)\,\frac{\left(\frac{1}{DF} - \frac{1}{10}\right)}{\left(\frac{1}{40} - \frac{1}{10}\right)}$$

$$= 10.04 + \frac{(-2.73)}{\left(-\frac{3}{40}\right)}\left(\frac{1}{DF} - \frac{1}{10}\right)$$

$$= 10.04 + \frac{36.4}{DF} - 3.64$$

$$= 6.40 + \frac{36.4}{DF}$$

G. MISCELLANEOUS

II Interpolation

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INTERPOLATION

A general purpose interpolation program is available for linear interpolation of any data in DRD standard format. The program does not extrapolate and, if any breaks in the input times occur, will not interpolate in these breaks. (The time interval which defines a break in the input times is specified by the user in the load card information). The first output time specified must be equal to or greater than the start time of the input, and the output rate of time, a constant Δt, must be specified. This output rate may be equal to, greater or less than the input data rate.

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G. MISCELLANEOUS

III Variate-Differences

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VARIATE DIFFERENCES

Introduction

The variate difference technique is a method of estimating the variance of the random element in a time-series by use of successive differences. It assumes that the non-random component of the series can be represented by a polynomial. Then the successive differencing of the series will gradually eliminate this polynomial content, but will not eliminate the random component. After the polynomial component becomes negligible, the variance of the random element remaining is estimated from the differenced data.

Mathematical Discussion

Let the time-series X(t) whose variance is to be estimated consist of N values of $\{t_i, X_i\}$, i=1, 2, $\cdots$, N. If it is assumed that X(t) can be approximated by a j^{th} degree polynomial

$$Y(t) = a_o + a_1 t + a_2 t^2 + \cdots + a_j t^j$$

then the variance of the data (σ_X^2) can be estimated from the variance of the residuals (σ_Δ^2), where

$$\Delta_t = [X(t) - Y(t)].$$

If Y(t) is a $(j-1)^{th}$ degree polynomial passing through the j points $X_{t-1}, X_{t-2}, \cdots, X_{t-j}$, it may be expressed as

$$Y_{t+i} = a_o + a_1 i + a_2 i^2 + \cdots + a_{(j-1)} i^{(j-1)}.$$

This polynomial, evaluated at point X_t, becomes simply $Y_t = a_o$.

It can be shown that the difference

$$\Delta_t = X_t - Y_t$$

is given by the j^{th} variate difference at point t.

The variance of these residuals can be found from

$$\sigma_\Delta^2 = \frac{1}{N-j-1} \sum_{i=1}^{N-j} (\Delta_i - \overline{\Delta})^2$$

where $\overline{\Delta}$ is the average of the residuals.

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If the $(j-1)^{th}$ polynomial is the best-fitting curve for the given data, $\overline{\Delta}$ will approach zero and the degrees of freedom (N-j-1) will be increased by one.

Then:

$$\sigma_{\Delta}^2 = \frac{1}{N-j} \sum_{i=1}^{N-j} \Delta_i^2$$

Since the Δ's are functions of the X's, the relationship between σ_{Δ}^2 and σ_X^2 can be expressed as

$$\sigma_{\Delta}^2 = \sum_{i=t-j}^{i=t} \left(\frac{\partial \Delta}{\partial x_i}\right)^2 \sigma_{x_i}^2$$

or, since $\sigma_{x_i}^2 = \sigma_{x_j}^2 \approx \sigma_x^2$

and

$$\sigma_{\Delta_t}^2 = \sigma_{\Delta_{t+1}}^2 = \sigma_{\Delta}^2$$

then

$$\sigma_{\Delta}^2 = \sigma_x^2 \sum_{i=t-j}^{i=t} \left(\frac{\partial \Delta}{\partial x_i}\right)^2$$

For the j^{th} order of differences

$$\sum_{i=t-j}^{i=t} \left(\frac{\partial \Delta}{\partial x_i}\right)^2 = \frac{(2j)!}{(j!)^2}$$

Thus $\sigma_{\Delta}^2 = \sigma_x^2 \left(\frac{(2j)!}{(j!)^2}\right)$

Hence, the variance of the data, σ_x^2, is found from

$$\sigma_x^2 = \sigma_{\Delta}^2 \left(\frac{(j!)^2}{(2j)!}\right)$$

or

$$\sigma_x^2 = \frac{(j!)^2}{(2j)!} \frac{\sum_{i=1}^{N-j} (\Delta_i)^2}{N-j}$$

Computer Program

The computer program will accept a maximum of 500 data points in one interval. Up to 10th variate differences may be computed for a maximum of 10 data fields in any run.

Table 1 gives the values of $\frac{(j!)^2}{(2j)!}$ for j=1 thru 10.

TABLE 1

j	$\frac{(j!)^2}{(2j)!}$	j	$\frac{(j!)^2}{(2j)!}$
1	1/2	6	1/924
2	1/6	7	1/3432
3	1/20	8	1/12,870
4	1/70	9	1/48,620
5	1/252	10	1/184,756

Numerical Example

Assume that the series $\{t_i, X_i\}$ can best be approximated by a 2nd degree polynomial

$$Y(t) = a_0 + a_1 t + a_2 t^2.$$

The coefficients a_0, a_1, a_2 can be found by fitting the curve to the points $(-1, X_{-1})$, $(-2, X_{-2})$ and $(-3, X_{-3})$, using the equations:

$$X_{-3} = a_0 - 3a_1 + 9a_2$$

$$X_{-2} = a_0 - 2a_1 + 4a_2$$

$$X_{-1} = a_0 - a_1 + a_2$$

The coefficients are given by:

$$a_0 = 3X_{-1} - 3X_{-2} + X_{-3}$$

$$a_1 = (5X_{-1} - 8X_{-2} + 3X_{-3})/2$$

$$a_2 = (X_{-1} - 2X_{-2} + X_{-3})/2$$

The polynomial evaluated at the point t=0 becomes

$$Y(t) = 3X_{-1} - 3X_{-2} + X_{-3}$$

By constructing a difference table it is easily seen that the third differences (since Y(t) is a 2nd degree curve) will be equal to

$$\Delta_t = (X_t - Y_t) = (X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3}).$$

DIFFERENCE TABLE (1)

Data	1st Differences	2nd Differences	3rd Differences
X_1	--	--	--
X_2	$X_2 - X_1$	--	--
X_3	$X_3 - X_2$	$X_3 - 2X_2 + X_1$	--
X_4	$X_4 - X_3$	$X_4 - 2X_3 + X_2$	$X_4 - 3X_3 + 3X_2 - X_1$
X_5	$X_5 - X_4$	$X_5 - 2X_4 + X_3$	$X_5 - 3X_4 + 3X_3 - X_2$
⋮	⋮	⋮	⋮
X_n	$X_n - X_{n-1}$	$X_n - 2X_{n-1} + X_{n-2}$	$X_n - 3X_{n-1} + 3X_{n-2} - X_{n-3}$

The data X_i, i=1 to 12, are as given in the table below. The sets of 1st, 2nd, 3rd, and 4th differences have been computed and also entered in the table.

i	X_i	1st Δ's	2nd Δ's	3rd Δ's	4th Δ's
1	4	--	--	--	--
2	9	5	--	--	--
3	19	10	5	--	--
4	28	9	-1	-6	--
5	42	14	5	6	12
6	55	13	-1	-6	-12
7	69	14	1	2	+8
8	88	19	5	4	2
9	106	18	-1	-6	-10
10	130	24	6	7	13
11	155	25	1	-5	-12
12	179	24	-1	-2	3

For each set of differences, compute a variance estimate for the Δ's,

$$\sigma_\Delta^2 = \frac{1}{N-j} \sum_{i=1}^{N-j} \Delta_i^2 .$$

and a variance estimate for the data, $\sigma_X^2 = \frac{(j!)^2}{(2j)!} \sigma_\Delta^2$.

Differences	σ_{Δ}^2	$\cdot \frac{(j!)^2}{(2j)!}$	$\sigma_{\hat{x}}^2$
1	293.54	1/2	146.77
2	11.70	1/6	1.95
3	26.88	1/20	1.344
4	97.25	1/70	1.389

Since the variance estimate obtained from the set of third differences is the minimum, we conclude that the data is best described by a 2nd degree polynomial, and that the variance estimate of the random component of the data is $\sigma_{\hat{x}}^2 = 1.344$.

VARIATE DIFFERENCES REFERENCES

Beers, Yardley. INTRODUCTION TO THE THEORY OF ERROR Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1957

Kendall, M.G. THE ADVANCED THEORY OF STATISTICS, Vol. II New York: Hafner Publishing Co. 1952

G. MISCELLANEOUS

IV Radar Cross-Section I (AGC and A-scope)

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RADAR CROSS SECTION

Introduction

The radar cross-section (σ) is the area intercepting that amount of power which, when scattered isotropically, produces an echo equal to that observed from the target. The radar cross-section of a target varies with the aspect angle (except in the case where the target is a sphere) and also with the frequency (wavelength) of the radar. Radar cross-section is, in general, not a simple function of geometrical cross-section, and hence must be determined empirically rather than by theoretical calculations.

Definitions of Symbols

σ = Radar Cross-section

σ_t = Radar cross-section of target (missile)

σ_S = Radar cross-section of calibration sphere

P_T = Power transmitted by the radar

P_R = Power received by the radar

P_{Rt} = Power received from target at Range R_t

P_{Rs} = Power received from sphere at Range R_s

P'_{Rs} = Equivalent power received from sphere at target Range R_t

G = Gain of radar antenna

λ = Wavelength of radar transmission

R = Slant range from radar to target

A_R = Effective antenna capture area

$\bar{B}$ = Constant obtained from sphere calibration

Mathematical Discussion

The basic radar equation used in the cross-section reduction can be derived in the following way: Assume that the radar antenna is omnidirectional, that is, it transmits power uniformly in all directions. Then, the power density (or power per unit area) at a distance R is equal to the transmitted power divided by the surface area of a sphere of radius R:

$$\text{Power density at R (omnidirectional antenna)} = \frac{P_T}{4\pi R^2}$$

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Since most radars employ directive antennas, the power density at any distance R will vary from that of an omnidirectional antenna by a factor G, called the "gain" of the antenna in the direction in question. Thus:

$$\text{Power density at R (directive antenna)} = \frac{P_T\, G}{4\pi R^2}$$

The total power incident upon the radar target at distance R equals the product of power density and target cross-section area (σ).

$$\text{Power incident on target} = \left(\frac{P_T\, G}{4\pi R^2}\right)\sigma$$

This incident power is then re-radiated in all directions. The re-radiated power density at the distance R (for example, re-radiated power density back at the radar receiving antenna) equals the incident power at the target divided by the surface area of a sphere of radius R.

$$\text{Power density at receiving antenna} = \left(\frac{P_T\, G}{4\pi R^2}\right)\frac{\sigma}{4\pi R^2}$$

The signal strength received by the radar antenna (P_R) is determined by the power density at the antenna and the effective antenna capture area (A_R).

$$P_R = \left[\left(\frac{P_T\, G}{4\pi R^2}\right)\left(\frac{\sigma}{4\pi R^2}\right)\right]A_R$$

Antenna theory provides the relationship between antenna gain and effective antenna area:

$$G = \frac{4\pi A_R}{\lambda^2} \text{ where } \lambda \text{ is the wavelength of the radiation}$$

or

$$A_R = \frac{G\lambda^2}{4\pi}$$

Thus, the power received by the radar is given by

$$P_R = \left(\frac{P_T\ G}{4\pi R^2}\right) \left(\frac{\sigma}{4\pi R^2}\right) \left(\frac{G\lambda^2}{4\pi}\right)$$

or $$P_R = \frac{P_T\ G^2\ \lambda^2\ \sigma}{(4\pi)^3\ R^4} \qquad (1)$$

This is one of the more common forms of the "radar equation".*

Solving (1) for target cross-section (σ), we find:

$$\sigma = \frac{(4\pi)^3\ R^4\ P_R}{P_T\ G^2\ \lambda^2} \qquad (2)$$

Since for any given radar P_T, G, and λ should be constant, equation (2) may be written:

$$\sigma = K\ R^4\ P_R$$

where $$K = \frac{(4\pi)^3}{P_T\ G^2\ \lambda^2} \qquad (3)$$

Because of the difficulties of measuring P_T, G, and λ, a simple method of determining K has been developed, by comparison of the target data with data obtained by tracking a sphere of known radius under identical conditions.

Thus, using the subscript t for target (missile) and s for the calibration sphere, we have:

$$\frac{\sigma_t}{\sigma_s} = \frac{K\ R_t^4\ P_{Rt}}{K\ R_s^4\ P_{Rs}}$$

$$\sigma_t = \left(\frac{P_{Rt}}{P_{Rs}}\right) \left(\frac{R_t}{R_s}\right)^4 \sigma_s \qquad (4)$$

*The radar equation as derived above and as used in the cross-section reduction is based upon the assumption of free-space transmission of electromagnetic radiation. No attempt has been made as yet to introduce corrections for atmospheric effects.

Equation (4) becomes much simplified if the ratio $\frac{R_t}{R_s}$ equals one; therefore it is desirable to derive an expression for P'_{Rs}, the power which would be received from the sphere of cross-section σ_s if it were at the same range as the target.

Since $\sigma_s = K\ R_s^4\ P_{Rs}$,

$$P_{Rs} = \frac{\sigma_s}{K\ R_s^4} \tag{5}$$

Since P_{Rs} is commonly measured not in watts but in decibels (db), Equation (5) becomes, by definition*

$$P_{Rs}\ (db) = 10 \log_{10}\left(\frac{\sigma_s}{K\ R_s^4}\right)$$

$$= 10 \log_{10} \sigma_s - 10 \log_{10} K - 40 \log_{10} R_s$$

or, since σ_s and K are constant for any given radar calibration,

$$P_{Rs}\ (db) = -40 \log_{10} R_s + B. \tag{6}$$

The received power data (P_{Rs} (db)) from the sphere calibration track may be evaluated at several points R_s, and the pairs of values R_{si}, P_{Rsi} (db), (i=1 to n), substituted in equation (6) to obtain n values of B_i. Since these may vary somewhat because of random fluctuations inherent in physical measuring systems, an average $\overline{B}$ is computed.

$$\overline{B} = \frac{\sum_{i=1}^{n} B_i}{n}$$

Thus, using equation (6), with $R_s = R_t$, the equivalent power P'_{Rs} is computed.

$$P'_{Rs}(db) = -40 \log_{10} R_t + \overline{B} \tag{7}$$

Equation (4) now becomes

$$\sigma_t = \frac{P_{Rt}}{P'_{Rs}} \sigma_s$$

*Definition of db: Quantity (db) = $10 \log_{10}\left(\frac{\text{Quantity}}{\text{reference Quantity}}\right)$

or again, since P_{Rt} and P'_{Rs} are measured in db:

$$\sigma_t(db) = P_{Rt}(db) - P'_{Rs}(db) + 10 \log_{10} \sigma_s \tag{8}$$

In order to convert σ_t(db) into units of area, the reference area is usually taken as one square meter (if σ_s was computed in square meters).

Thus

$$\sigma_t \ (m^2) = \frac{1}{10} \text{ antilog}_{10} \ (\sigma_t \ (db)) \tag{9}$$

Computer Program

Since cross-section varies with aspect angle and since range data is needed in computing σ, the cross-section reduction program must have as part of its input the radar output tape from the DRD Velocity and Acceleration program (discussed separately). The other necessary inputs are a digital tape of the "power received vs time" data, the radar coordinates in the same system as the "V and A" data, the cross-section (σ_s) of the calibration sphere used, and a hand-computed value of the $\overline{B}$ of equation (7).

The program produces a listing of the following functions, vs time:

slant range (in feet, meters or yards)

altitude (feet or meters)

aspect angle (degrees)

cross-section (square meters)

cross-section (db above 1 square meter)

$\log_{10}$ (range) (feet or meters)

power received from target (db)

The altitude listed is the same as that computed by the V and A program. The aspect angle is defined as the angle between the target velocity vector and the line of sight to the radar.

The "power received" data input to the reduction may be recorded originally in any one of three ways:

1. The radar automatic gain control (AGC) signal, a measure of received power strength, may be recorded in analog form on magnetic tape, and digitized by the DRD telemetry station digitizer.

2. The received radar pulses may be displayed on an A-scope and photographed by either strip or frame cameras with timing, and the received power measurements (peak amplitude of the pulses) read on a Telereadex (front projector). The card output of the reader is then transferred to digital magnetic tape in the same format as that of the telemetry digitizer output.

3. The AGC signal may be recorded on Sanborn strip recorders, which must be read manually. These can also be digitized on the Telereadex, and the data processed as above.

Once the "power received" data has been converted to its digital tape format (the same format used for digitized FM telemetry data), processing is identical for all three types of raw data.

RADAR CROSS-SECTION REFERENCES

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II. ADDENDUMS

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H. ADDENDUMS

I Rotations of Cartesian Coordinate Systems

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ADDENDUM TO

HANDBOOK OF DATA REDUCTION METHODS

ROTATIONS OF CARTESIAN COORDINATE SYSTEMS

24 September 1964

JOSEPH H. CORSON DRD-T, WSMR

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ABSTRACT

The matrix equation used to rotate position data from one left-handed Cartesian system to another is developed by successive rotations about the coordinate axes through the geodetic positions of the origins of the two systems. A final rotation then references the system to the line of fire of the missile. The matrix to accomplish these rotations in one step is derived in this report, and its inverse solution is also presented.

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ROTATIONS OF CARTESIAN COORDINATE SYSTEMS

Introduction:

This report derives the matrix equation used to rotate position data from one left-handed Cartesian system to another left-handed Cartesian system. The matrix is developed by successive rotations about the coordinate axes through the geodetic positions of the origins of the two systems and through a final rotation to reference the system to the line of fire of the missile. The inverse solution is also derived.

The Cartesian systems to be considered here are defined as coordinate systems (x, y, z) in which x and y lie in a plane parallel to the plane tangent to Clarke's Spheroid of 1866 at the origins. The z coordinates are perpendicular to the xy planes and positive upwards. The original Cartesian system is oriented on true North.

Mathematical Derivation:

Definitions of symbols used in the derivation:

ϕ_0, λ_0 — geodetic positions for the original origin.

ϕ_t, λ_t — geodetic positions for the new origin.

ϕ_0 and ϕ_t are positive in the northern hemisphere.

λ_0 and λ_t are positive in the western hemisphere.

α — azimuth of fire[1] - the azimuth to which the new system is oriented and measured positive clockwise from north.

x_t, y_t, z_t — Cartesian coordinates of the new origin; x_t positive north, y_t positive east, z_t positive upwards and perpendicular to the x_t y_t plane.

x, y, z — Cartesian coordinates of a point to be rotated to the new Cartesian system.

x', y', z' — the coordinates of the point in the new system; x' positive along the azimuth in the new reference plane, y' positive to the right of the azimuth in the new reference plane, and z' positive upwards and perpendicular to the new reference plane.

$\Delta x = x - x_t$; $\Delta y = y - y_t$; $\Delta z = z - z_t$.

[1]Azimuth of fire in this report refers to true azimuth which is measured from the meridian of the new origin, 0 degrees being North.

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Figure 1 shows the tangent planes of the origins of the two systems.

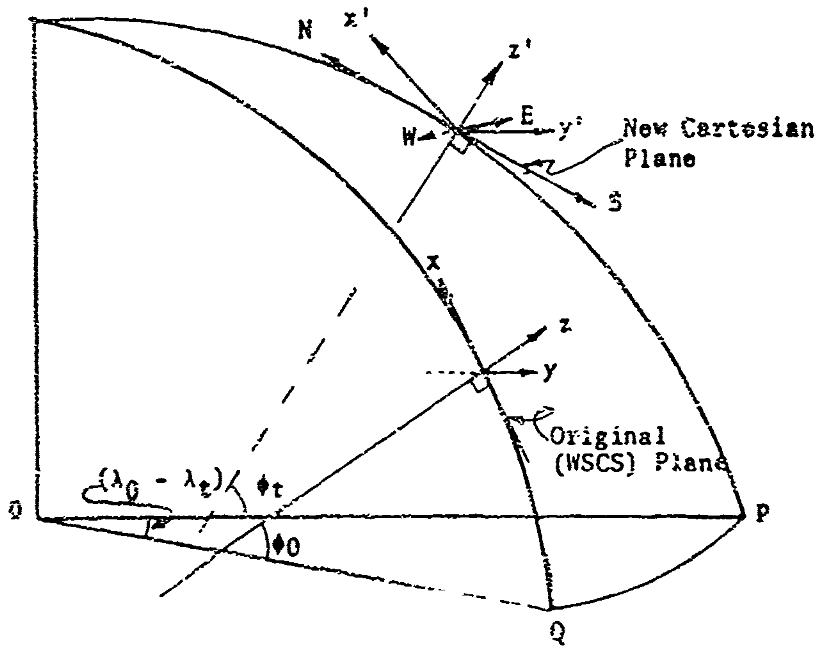
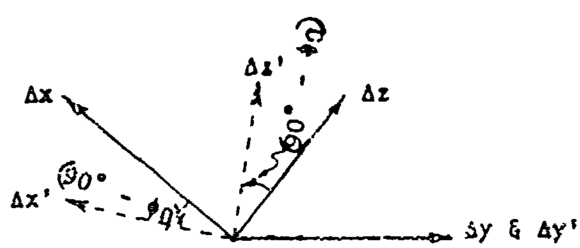


FIGURE 1

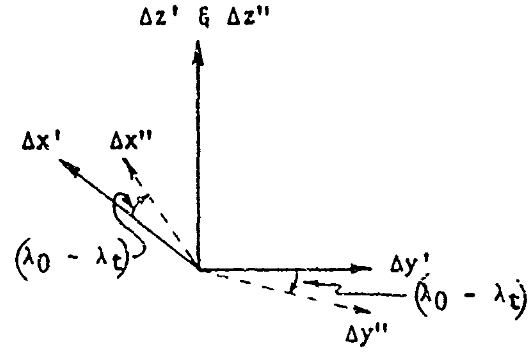
OPQ represents the equatorial plane.

The first rotation is about the y axis, through an angle of $(90° - \phi_0)$, to a plane parallel to the equatorial plane.



$$
\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \sin \phi_0 & 0 & -\cos \phi_0 \\ 0 & 1 & 0 \\ \cos \phi_0 & 0 & \sin \phi_0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} . \qquad (1)
$$

Then rotate about the z' axis through the angle $(\lambda_0 - \lambda_t)$.

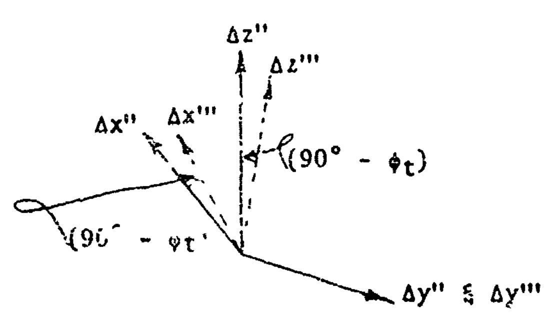


$$
\begin{bmatrix} \Delta x'' \\ \Delta y'' \\ \Delta z'' \end{bmatrix} = \begin{bmatrix} \cos (\lambda_0 - \lambda_t) & -\sin (\lambda_0 - \lambda_t) & 0 \\ \sin (\lambda_0 - \lambda_t) & \cos (\lambda_0 - \lambda_t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} . \qquad (2)
$$

Combining equations (1) and (2) yields the following matrix form.

$$
\begin{bmatrix} \Delta x'' \\ \Delta y'' \\ \Delta z'' \end{bmatrix} = \begin{bmatrix} \sin \phi_0 \cos (\lambda_0 - \lambda_t) & -\sin (\lambda_0 - \lambda_t) & -\cos \phi_0 \cos (\lambda_0 - \lambda_t) \\ \sin \phi_0 \sin (\lambda_0 - \lambda_t) & \cos (\lambda_0 - \lambda_t) & -\cos \phi_0 \sin (\lambda_0 - \lambda_t) \\ \cos \phi_0 & 0 & \sin \phi_0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} . \qquad (3)
$$

Rotate about the y" axis through an angle $(90 - \phi_t)$, such that the reference plane is tangent to the new origin.



$$
\begin{bmatrix} \Delta x''' \\ \Delta y''' \\ \Delta z''' \end{bmatrix} = \begin{bmatrix} \sin\phi_t & 0 & \cos\phi_t \\ 0 & 1 & 0 \\ -\cos\phi_t & 0 & \sin\phi_t \end{bmatrix} \begin{bmatrix} \Delta x'' \\ \Delta y'' \\ \Delta z'' \end{bmatrix} . \tag{4}
$$

Equations (3) and (4) yield the following matrix form.

$$
\begin{bmatrix} \Delta x''' \\ \Delta y''' \\ \Delta z''' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{5}
$$

where

$$a_{11} = \sin\phi_0 \sin\phi_t \cos(\lambda_0 - \lambda_t) + \cos\phi_0 \cos\phi_t$$

$$a_{12} = -\sin\phi_t \sin(\lambda_0 - \lambda_t)$$

$$a_{13} = -\cos\phi_0 \sin\phi_t \cos(\lambda_0 - \lambda_t) + \sin\phi_0 \cos\phi_t$$

$$a_{21} = \sin\phi_0 \sin(\lambda_0 - \lambda_t)$$

$$a_{22} = \cos(\lambda_0 - \lambda_t)$$

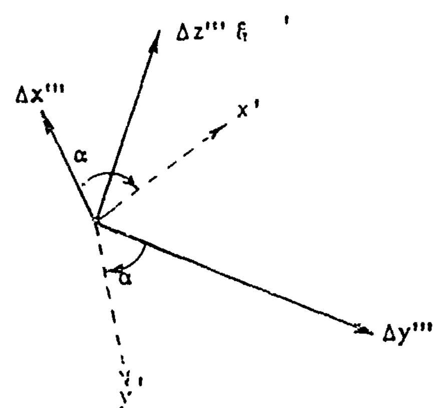
$$a_{23} = -\cos\phi_0 \sin(\lambda_0 - \lambda_t)$$

$$a_{31} = -\sin\phi_0 \cos\phi_t \cos(\lambda_0 - \lambda_t) + \sin\phi_t \cos\phi_0$$

$$a_{32} = \cos\phi_t \sin(\lambda_0 - \lambda_t)$$

$$a_{33} = \cos\phi_0 \cos\phi_t \cos(\lambda_0 - \lambda_t) + \sin\phi_0 \sin\phi_t .$$

The final rotation is about the z''' axis through an angle, α, which orients the system to the line of fire.



$$
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x''' \\ \Delta y''' \\ \Delta z''' \end{bmatrix}. \tag{6}
$$

Substituting equation (5) in equation (6) yields

$$
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{7}
$$

where

$$
\begin{aligned}
c_{11} &= \sin\phi_0 \sin\phi_t \cos\alpha \cos(\lambda_0 - \lambda_t) + \cos\phi_0 \cos\phi_t \cos\alpha \\
&\quad + \sin\phi_0 \sin\alpha \sin(\lambda_0 - \lambda_t)
\end{aligned}
$$

$$
c_{12} = -\sin\phi_t \cos\alpha \sin(\lambda_0 - \lambda_t) + \sin\alpha \cos(\lambda_0 - \lambda_t)
$$

$$
\begin{aligned}
c_{13} &= -\cos\phi_0 \sin\phi_t \cos\alpha \cos(\lambda_0 - \lambda_t) + \sin\phi_0 \cos\phi_t \cos\alpha \\
&\quad - \cos\phi_0 \sin\alpha \sin(\lambda_0 - \lambda_t)
\end{aligned}
$$

$$
\begin{aligned}
c_{21} &= -\sin\phi_0 \sin\phi_t \sin\alpha \cos(\lambda_0 - \lambda_t) - \cos\phi_0 \cos\phi_t \sin\alpha \\
&\quad + \sin\phi_0 \cos\alpha \sin(\lambda_0 - \lambda_t)
\end{aligned}
$$

$$
c_{22} = + \sin\phi_t \sin\alpha \sin(\lambda_0 - \lambda_t) + \cos\alpha \cos(\lambda_0 - \lambda_t)
$$

$$
\begin{aligned}
c_{23} &= \cos\phi_0 \sin\phi_t \sin\alpha \cos(\lambda_0 - \lambda_t) - \sin\phi_0 \cos\phi_t \sin\alpha \\
&\quad - \cos\phi_0 \cos\alpha \sin(\lambda_0 - \lambda_t)
\end{aligned}
$$

$$
c_{31} = -\sin\phi_0 \cos\phi_t \cos(\lambda_0 - \lambda_t) + \sin\phi_t \cos\phi_0
$$

$$
c_{32} = \cos\phi_t \cdot \sin(\lambda_0 - \lambda_t)
$$

$$
c_{33} = \cos\phi_0 \cos\phi_t \cos(\lambda_0 - \lambda_t) + \sin\phi_0 \sin\phi_t .
$$

Making the following substitution in equation (7)

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{8}$$

gives the matrix equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [C] \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{9}$$

where x' is positive along the line of fire, y' is positive to the right of x' and z' is positive upwards and perpendicular to the x' y' plane.

The reverse solution is found from the matrix equation

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = [C]^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{10}$$

The determinant of the matrix C is

$$|C| = 1$$

therefore, equation (10) is an orthogonal transformation or a transformation of rotation. In an orthogonal transformation the inverse matrix $[C]^{-1}$ is equal to the transpose, $[C]'$, of $[C]$.

Therefore

$$[C]^{-1} = [C]' = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \tag{11}$$

The most common usages of these rotations are to rotate from the White Sands Cartesian Coordinate system to a launcher tangent Cartesian system or from a local tangent plane to WSCS. In the WSCS system the origin is the intersection of latitude, ϕ_0, 33°05'00" north and longitude, λ_0, 106°20'00" west. At this point the EN-plane is tangent to the surface of the Clarke Spheroid of 1866. The E-axis is an east-west line tangent to the Clarke Spheroid at the origin, positive in the eastward direction. The N-axis is a north-south line lying on the meridian and tangent to the Clarke Spheroid at the origin, positive in the northward direction. The z-axis is perpendicular to the EN-plane and positive upwards.

The WSCS origin has been given an arbitrary value of E = 500,000.00 feet and N = 500,000.00 feet. These values were selected so as to cause the E and N coordinates to be positive within the limits of the range.

Geodetic azimuth is measured from the meridian of the new origin, zero degrees being south. Since true azimuth is geodetic azimuth ±180°, it is measured off the meridian of the new origin, 0 degrees being north. A transverse Mercator azimuth (WSTM or UTM) is measured from a line that passes through the new origin and is parallel to the central meridian of the transverse Mercator system. The azimuth is measured with 0 degrees being transverse Mercator north.

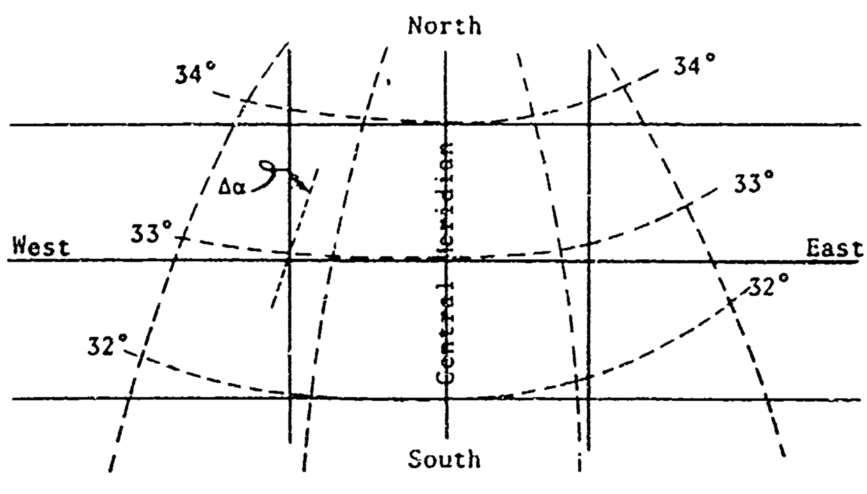


FIGURE 3.

Figure 3 illustrates the geodetic lines and transverse Mercator lines. The solid lines designate UTM or WSTM grid lines. The dotted lines show longitude and latitude. $\Delta\alpha$ is the difference between geodetic north and grid north.

The azimuth of fire (α) used in the previous equations is the true azimuth. If a transverse Mercator azimuth (α_{TM}) is given, the true azimuth may be found from the following equation.

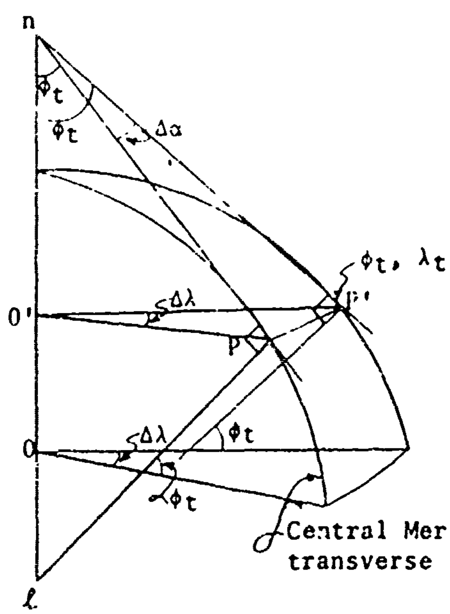
$$\alpha = \alpha_{TM} + 2 \sin^{-1} \left(\sin \phi_t \sin \frac{\Delta\lambda}{2} \right)$$

where

$\Delta\lambda$ = longitude of the central meridian of the transverse Mercator system less the longitude of the new origin (λ_t).

ϕ_t = latitude of the new origin.

The following diagram describes the correction ($\Delta\alpha$) to be added to the transverse Mercator azimuth to obtain the true azimuth.



$$O'P = O'P' = nP \sin \phi_t$$

$$PP' = 2O'P \sin \frac{\Delta\lambda}{2}$$

$$= 2nP \sin \phi_t \sin \frac{\Delta\lambda}{2}$$

$$2 \sin \frac{\Delta\alpha}{2} = \frac{PP'}{nP}$$

$$= 2 \sin \phi_t \sin \frac{\Delta\lambda}{2}$$

$$\Delta\alpha = 2 \sin^{-1} \left[\sin \phi_t \sin \frac{\Delta\lambda}{2} \right]$$

FIGURE 4.

For small differences of longitude, ($\Delta\lambda \sin \phi_t$) may be substituted for

$2 \sin^{-1} \left(\sin \phi_t \sin \frac{\Delta\lambda}{2} \right)$, therefore the equation for true azimuth becomes:

$$\alpha = \alpha_{TM} + \Delta\lambda \sin \phi_t.$$

II. ADDENDUMS

II Power Spectral Density Analysis Using the ALL Analyzer

ADDENDUM NO. 2
to
THE HANDBOOK OF DATA REDUCTION

POWER SPECTRAL DENSITY ANALYSIS
USING THE AEL ANALYZER

December 1964

Prepared by: Virginia B. Tipton
VIRGINIA B. TIPTON

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POWER SPECTRAL DENSITY ANALYSIS
USING THE AEL ANALYZER

Introduction

The AEL Spectrum Analyzer was built by American Electronic Laboratories, Inc., of Colmar, Pennsylvania. The analyzer is designed to separate a complex signal into its various frequency components and determine the relative magnitudes or power levels of each of these components. The analyzer consists basically of 3 banks of 20 filters, each having a different center frequency and bandwidth, a set of integrators which sum the continuous outputs of each of the filters, and a commutated readout which samples, digitizes and records the data from each integrator in turn. The complete set of 60 filters covers the entire band from 10 cps to 10 KC.

Two sets of white noise calibration signals may be passed thru the analyzer in addition to the data signal: The first, white noise machine calibration, is necessary to compute a correction factor for each filter to compensate for any nonlinearity in filter gain adjustments. The second, another white noise input which is related to the data by known factors, is necessary if the digital output of the analyzer is to be calibrated in the physical units of the input data.

The computer program is required to accept the raw data input, compute and apply the necessary correction factors and calibration constants, and generate several types of data output in formats suitable for a variety of listing and plotting requirements.

Derivation of Relationships

The spectrum analyzer is a device which receives a complex signal as input and separates the signal into its various frequency components. This analyzer consists of three banks of twenty filters each, the first bank of 20 filters covering the frequency range of 10 to 100 cycles per second, the second bank 100 cps to 1 KC, and the third 1 KC to 10 KC. Only one bank of filters may be used in a single pass of the input signal. In order to cover the complete frequency range of the analyzer, three passes of the input signal must be made. A commutated readout samples each of the twenty filters once each second or tenth of a second. A frame of data consists of time and these twenty digitized readouts. The last readout of a pass is called last frame data, i.e., there will be 3 last frames for one complete analysis.

Two white noise calibrating signals are passed through the analyzer in addition to the data signal. (White noise is a random signal which has uniform power distribution throughout the frequency range of the analyzer.) The bandwidths of the filters of the analyzer are so designed that, when white noise is processed, the output of each filter will have 0.5 db gain over the output of the filter which preceeded it. This characteristic is used to compute a correction factor for fine adjustment of the analyzer. If the machine is out of adjustment, the actual gain of each filter will not equal the predicted gain, in which case a correction factor is computed for the filter which is in error. Three banks of white noise data are processed, utilizing all 60 filters, but only the last frames of the white noise calibrations are recorded on the data tape.

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The second calibration signal is also a white noise (data) signal. This signal has known relational factors between it and a given function value, thus allowing the data signal to be calibrated in the proper physical units. Only the 3 last frame outputs of the calibration are recorded.

A "group" of data consists of three banks of last frame white noise (machine) calibrations, three banks of last frame white noise (data) calibrations associated with the data, and N frames of data for each of the three banks for the period of analysis. Sometimes reduction of only the last frame data will be requested.

"Pot" Factor Calculations

The input voltage to the analyzer is attenuated by a potentiometer. When switching from one bank of filters to another it is usually necessary to change the potentiometer (pot) setting to achieve optimum output. In order to relate all banks of information to each other the data from each bank must be multiplied by a pot factor. This means that there will be three white noise (machine) pot factors (one for each bank), three white noise (data) pot factors, and three pot factors for data to be computed. The pot factor to be used for any bank B is the square of one over the pot setting for that bank of data. The pot settings used will be supplied by the program requestor.

$$P.F. = \left(\frac{1}{P.S.}\right)^2$$

White Noise Correction Factors

The white noise (machine) correction factor for each filter compensates for fine adjustment of the analyzer. The analyzer is designed to have 0.5 db gain between any two adjacent filters when processing white noise. The correction factors are calculated using the white noise (machine) calibrations multiplied by their appropriate pot factors.

Let $M_j = (WN_j)(PF_B)$ be the white noise (machine) data output of the j^{th} filter corrected by the pot factor for bank B, where

$B = 1$ for filter numbers $1 \leq j \leq 20$

$B = 2$ for filters $21 \leq j \leq 40$

$B = 3$ for filters $41 \leq j \leq 60$.

Then the 0.5 db gain criterion between adjacent filters leads to the following relationships for filter correction factors, R_j.

$$10 \log_{10}\left(\frac{R_2 M_2}{R_1 M_1}\right) = 0.5 \text{ db}$$

$$10 \log_{10}\left(\frac{R_3 M_3}{R_2 M_2}\right) = 0.5 \text{ db}$$

$$\vdots$$

$$10 \log_{10}\left(\frac{R_j M_j}{R_{j-1} M_{j-1}}\right) = 0.5 \text{ db}.$$

Using filter number 1 as a reference and solving for the correction factors R_j, $(j = 1, 2, \cdots, 60)$, leads to the equations:

$R_1 = 1$, by definition,

$$R_2 = 10^{(.05)}\left(\frac{M_1}{M_2}\right)$$

$$R_3 = 10^{(.05)}\left(\frac{M_2}{M_3}\right)R_2 = 10^{(.05)}\left(\frac{M_2}{M_3}\right)10^{(.05)}\left(\frac{M_1}{M_2}\right) = 10^{(.05)(2)}\left(\frac{M_1}{M_3}\right)$$

$$R_4 = 10^{(.05)}\left(\frac{M_3}{M_4}\right)R_3 = 10^{(.05)}\left(\frac{M_3}{M_4}\right)10^{(.05)(2)}\left(\frac{M_1}{M_3}\right) = 10^{(.05)(3)}\left(\frac{M_1}{M_4}\right)$$

$$\vdots$$

$$R_j = 10^{(.05)(j-1)}\left(\frac{M_1}{M_j}\right).$$

Filter Bandwidth Factors

The different filters of the analyzer have different bandwidths. In order to properly compare the outputs of the various filters as a function per cycle it is necessary to normalize the data by dividing each filter's output by that filter's bandwidth. All normalized data are then in units of filter output per cycle of bandwidth

The bandwidth of the j^{th} filter is given by the difference between the lower band edge frequency (L) of the j^{th} filter and the lower band edge fre quency of the $(j+1)^{th}$ filter, where the lower band edge frequencies of adjacent filters are related by the 0.5 db criterion previously used in computing white noise correction factors,

That is,

$$BW_j = L_{(j+1)} - L_{(j)}$$

and

$$10 \log_{10}\left(\frac{L_{(j+1)}}{L_{(j)}}\right) = 0.5 \text{ db}$$

or

$$\log_{10}\left(L_{(j+1)}\right) - \log_{10}\left(L_{(j)}\right) = 0.05.$$

From this it can be shown that

$$\log_{10}(L_2) = 0.05 + \log_{10}(L_1)$$

$$\log_{10}(L_3) = 0.05 + \log_{10}(L_2) = 0.05 + 0.05 + \log_{10}(L_1)$$

$$\log_{10}(L_4) = 0.05 + \log_{10}(L_3) = 0.05 + 0.05 + 0.05 + \log_{10}(L_1)$$

$$= (3)(0.05) + \log_{10}(L_1)$$

$$\vdots$$

$$\log_{10}(L_j) = 0.05 + \log_{10}(L_{j-1}) = (0.05)(j-1) + \log_{10}(L_1).$$

Since $L_1 = 10$ cps and $\log_{10}(L_1) = 1.0$, this expression for the j^{th} lower band edge frequency can be written

$$\log_{10}(L_j) = (j-1)(0.05) + 1.0$$

or

$$L_j = 10^{[(j-1)(0.05) + 1]} = \left[10^{(j-1)(0.05)}\right](10).$$

Then the bandwidth of the j^{th} filter can be found from

$$BW_j = L_{(j+1)} - L_{(j)}$$

$$= 10\left[10^{(j)(.05)} - 10^{(j-1)(.05)}\right]$$

$$= 10\left[10^{(j-1)(.05)}\right]\left[10^{(.05)} - 1\right]$$

$$= \left[10^{(j-1)(.05)}\right]\left[10\ (1.122 - 1)\right]$$

$$= \left[10^{(j-1)(.05)}\right](1.22).$$

Data Calibration

In order to calibrate the data in some physical units, a calibration signal of known RMS value must be analyzed and digitized. The calibration furnished by the data user may be either a sine wave or a white noise signal; however, the calibration signal processed on the analyzer will always be a white noise signal. If a sine wave is furnished it will be examined to determine its RMS value and the output of a white noise generator, matched to this RMS value, will be fed to the analyzer for the calibration.

The following is the derivation of the calibration constant K which is used to convert machine count values of analyzer output to engineering units. First it will be shown that K is the same constant for every filter; then the formula for finding K will be derived. In practice, because of small variations in actual filter outputs, a separate K_j should be computed for each filter ($j = 1$ to 60) and the average value of all 60 K_j's used in the reduction for all filters, thus increasing the statistical reliability of the estimate.

A. Proof that K is the same constant for each filter, i.e., $K_1 = K_2 = \cdots = K_{60}$

The total output power of the noise calibration is the sum of the outputs of the 60 filters:

$$P_n = \sum_{j=1}^{60} P_{out_j}. \quad (1)$$

However, the output of each filter is given in terms of a relative machine count value,

$P_{out_j} = K_j D_j$, where K_j is the calibration constant for the j^{th} filter and D_j the machine count value.

The design characteristics of the analyzer are such that the power output relationship between filters is given by:

$$P_{out_1} = \frac{P_{out_j}}{10^{.05(j-1)}}. \quad (2)$$

The normalized power output (power/cycle) of any filter is given by

$$\frac{P_{out_1}}{BW_1} = \text{power/cycle} = \frac{P_{out_j}}{BW_1[10^{.05(j-1)}]} \tag{3}$$

or since $P_{out_j} = K_j D_j$

$$\frac{K_1 D_1}{BW_1} = \frac{K_j D_j}{BW_1[10^{.05(j-1)}]} . \tag{4}$$

Again utilizing the design characteristics of the analyzer it is seen that:

$$.5(j-1) = 10 \log\left(\frac{D_j}{D_1}\right) \tag{5}$$

or

$$D_1 = \frac{D_j}{10^{.05(j-1)}} . \tag{6}$$

Substituting equation (6) into equation (4) gives

$$\frac{K_1 D_j}{BW_1 [10^{.05(j-1)}]} = \frac{K_j D_j}{[10^{.05(j-1)}] BW_1} \tag{7}$$

or simply

$K_1 = K_j$ for all j.

B. Derivation of the Formula for Computing K

The total power output of the white noise generator can be divided into two fractional parts: that portion of the generator's frequency range which is analyzed by the PSD analyzer, and that portion of the noise generator's frequency range which is not analyzed by the analyzer.

That is:

$$P_N = \left(\frac{BW_A}{BW_N}\right)P_N + \left(\frac{A}{BW_N}\right)P_N \tag{8}$$

where

P_N = total power output of white noise generator

BW_A = Band width of PSD analyzer

BW_N = Band width of noise generator

$\left(\frac{BW_A}{BW_N}\right), \left(\frac{A}{BW_N}\right)$ = fractions of noise power analyzed and not analyzed respectively.

But that part of the noise generator's output which is analyzed is also the total power output of the 60 filters

$$\left(\frac{BW_A}{BW_N}\right)P_N = \sum_{j=1}^{60} P_{out_j}. \tag{9}$$

Substituting this, equation (8) becomes;

$$P_N = \sum_{j=1}^{60} P_{out_j} + \left(\frac{A}{BW_N}\right)P_N \tag{10}$$

By definition of white noise, which states that the normalized power per cycle output at any frequency in the band equals the total power output averaged over the total frequency band, we have

$$\frac{P_{out_j}}{BW_j} = \frac{\sum_{j=1}^{60} P_{out_j}}{BW_A} \tag{11}$$

But since $P_{out_j} = \frac{K_j D_j}{T}$*, (12)

equation (11) can be written

$$\frac{\sum_{j=1}^{60} P_{out_j}}{BW_A} = \frac{K_j D_j}{BW_j T} \tag{13}$$

where T is the period of integration,

Substituting equation (13) into equation (10) gives

$$P_N = \frac{K_j D_j BW_A}{BW_j T} + \left(\frac{A}{BW_N}\right)P_N \tag{14}$$

which can be solved for K;

$$K = P_N\left(1 - \frac{A}{BW_N}\right)\left[\frac{BW_j T}{BW_A D_j}\right]. \tag{15}$$

---

*The factor $\frac{1}{T}$ in equation (12) is the result of the fact that machine count output e analyzer is the read-out of an integrator, $D_j = K\int_0^T f(t)\,dt$, and is thus in units of "(power/cycle) x time". Since we wish to deal simply in normalized (power/cycle), this time factor must be compensated for.

For the white noise generator in the PSD analyzer the constants are:

$$BW_N = (27\ KC)\ \pi/2 \quad = 42.4\ KC$$

$$A = (BW_N - 10\ KC) = 32.4\ KC$$

$$BW_A = 10\ KC.$$

Substituting these values in (15),

$$K = P_N \left(1 - \frac{32.4}{42.4}\right)\left[\frac{BW_j\ T}{(10\ KC)\ D_j}\right]$$

$$= \left(2.35849 \times 10^{-4}\right)\frac{P_N\ T\ BW_j}{D_j} \qquad (16)$$

which can be evaluated from the white noise data calibration for any analysis group.

Data Output Types

A great variety of output data types may be requested from the program. However, for any given run only one data type will be requested. The form of the output requested may be listings, plots, or both.

For the sake of ease of reference, the 33 data types are numbered, and defined as below:

1. Data 1 is the digital data output of the analyzer corrected for a factor of "System Zero" and formatted as the input to the 7094 reduction program.

2. Data 2 is the most basic output data type. It is simply the raw data corrected for the appropriate pot factors and white noise (machine) calibration. Data $2_{j,n}$ is the relative power contained in the j^{th} filter bandwidth, summed over the time interval $(t_n - t_0)$.

$$\text{Data } 2_{j,n} = (\text{Data } 1_{j,n})(P.F._B)(R_j)$$

where $P.F._B$ is the pot factor for the bank of filters containing the j^{th} filter and R_j is the white noise (machine) correction factor for the j^{th} filter.

3. Data $3_{j,n}$ is the relative power per cycle contained in the j^{th} filter bandwidth summed over the time interval $(t_n - t_0)$ normalized by the width of the j^{th} filter in cycles. Data $3_{j,n} = \dfrac{\text{Data } 2_{j,n}}{BW_j}$.

4-5. The RMS values of the power measurements of data types 2 and 3 give the relative G's and relative G's per cycle outputs in the j^{th} filter in the time interval $(t_n - t_0)$.

$$\text{Data } 4_{j,n} = \sqrt{\text{Data } 2_{j,n}}$$

$$\text{Data } 5_{j,n} = \sqrt{\text{Data } 3_{j,n}} = \sqrt{\frac{\text{Data } 2_{j,n}}{BW_j}}$$

6-9. Data types 2-5 are total power or G measurements over the time intervals $(t_n - t_0)$ where $n = 1, 2, \cdots, N$ give the successive time points of analyzer output. Data types 6-9 are the average relative power or G's in the j^{th} filter bandwidth, averaged over the total integration time t_N.

$\text{Data } 6_{j,N} = \dfrac{\text{Data } 2_{j,N}}{(t_N - t_0)}$ where Data $2_{j,N}$ is the last frame data of the j^{th} filter and t_N is the time of last frame.

$$\text{Similarly, Data } 7_{j,N} = \frac{\text{Data } 3_{j,N}}{(t_N - t_0)} = \frac{\text{Data } 2_{j,N}}{(BW_j)(t_N - t_0)}$$

$$\text{Data } 8_{j,N} = \frac{\text{Data } 4_{j,N}}{(t_N - t_0)} = \frac{\sqrt{\text{Data } 2_{j,N}}}{(t_N - t_0)}$$

$$\text{Data } 9_{j,N} = \frac{\text{Data } 5_{j,N}}{(t_N - t_0)} = \frac{\sqrt{\text{Data } 3_{j,N}}}{(t_N - t_0)}$$

$$= \frac{1}{(t_N - t_0)} \sqrt{\frac{\text{Data } 2_{j,N}}{BW_j}}$$

10-13. Data types 10 thru 13 are derived from types 2-5 and the calibration constant K, which expresses the relative data in the proper physical units

$\text{Data } 10_{j,n} = K \text{ Data } 2_{j,n}$ is th[illegible] total power in the j^{th} bandwidth in the time interval $(t_n - t_0)$

$$\text{Data } 11_{j,n} = K \text{ Data } 3_{j,n} = \frac{K \text{ Data } 2_{j,n}}{BW_j}$$

$$\text{Data } 12_{j,n} = \sqrt{K} \text{ Data } 4_{j,n} = \sqrt{K \text{ Data } 2_{j,n}}$$

$$\text{Data } 13_{j,n} = \sqrt{K} \text{ Data } 5_{j,n} = \sqrt{K \text{ Data } 3_{j,n}} = \sqrt{\frac{K \text{ Data } 2_{j,n}}{BW_j}}$$

14-17. Data types 14 thru 17 are the calibrated last frame data averaged over the total integration time, $(T_N - T_0)$

$$\text{Data } 14_{j,N} = \text{K Data } 6_{j,N} = \frac{\text{K Data } 2_{j,N}}{(T_N - T_0)}$$

$$\text{Data } 15_{j,N} = \text{K Data } 7_{j,N} = \frac{\text{K Data } 2_{j,N}}{BW_j (T_N - T_0)}$$

$$\text{Data } 16_{j,N} = \sqrt{K} \text{ Data } 8_{j,N} = \frac{\sqrt{\text{K Data } 2_{j,N}}}{(T_N - T_0)}$$

$$\text{Data } 17_{j,N} = \sqrt{K} \text{ Data } 9_{j,N} = \frac{1}{(T_N - T_0)} \sqrt{\frac{\text{K Data } 2_{j,N}}{BW_j}}$$

Requests for data types 14 thru 17 will always be for last frame only reductions.

18-33. Data types 18 thru 33 are incremental data types showing the changes and rates of change of data from one point to the next in the analysis. When the time interval Δt between adjacent time points $(t_n - t_{n-1})$ is one second, the rates of change of data types 22-25 and 30-33 will be numerically equal to their respective incremental data types 18-21 and 26-29.

$$\text{Data } 18_{j,n} = \text{Data } 2_{j,n} - \text{Data } 2_{j,n-1}$$

$$\text{Data } 19_{j,n} = \frac{\text{Data } 2_{j,n} - \text{Data } 2_{j,n-1}}{BW_j}$$

$$\text{Data } 20_{j,n} = \sqrt{\text{Data } 2_{j,n}} - \sqrt{\text{Data } 2_{j,n-1}}$$

$$\text{Data } 21_{j,n} = \frac{1}{\sqrt{BW_j}} \left(\sqrt{\text{Data } 2_{j,n}} - \sqrt{\text{Data } 2_{j,n-1}} \right)$$

$$\text{Data } 22_{j,n} = \text{Data } 18_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 23_{j,n} = \text{Data } 19_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 24_{j,n} = \text{Data } 20_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 25_{j,n} = \text{Data } 21_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 26_{j,n} = K(\text{Data } 2_{j,n} - \text{Data } 2_{j,n-1})$$

$$\text{Data } 27_{j,n} = \frac{K}{BW_j}(\text{Data } 2_{j,n} - \text{Data } 2_{j,n-1})$$

$$\text{Data } 28_{j,n} = \sqrt{K}\left(\sqrt{\text{Data } 2_{j,n}} - \sqrt{\text{Data } 2_{j,n-1}}\right)$$

$$\text{Data } 29_{j,n} = \sqrt{\frac{K}{BW_j}}\left(\sqrt{\text{Data } 2_{j,n}} - \sqrt{\text{Data } 2_{j,n-1}}\right)$$

$$\text{Data } 30_{j,n} = \text{Data } 26_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 31_{j,n} = \text{Data } 27_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 32_{j,n} = \text{Data } 28_{j,n}/(t_n - t_{n-1})$$

$$\text{Data } 33_{j,n} = \text{Data } 29_{j,n}/(t_n - t_{n-1})$$

Summary of Data Types

Data 1 = Input Data

2 = $(D_1)(PF_B)(R_j)$	3 = $\frac{D_2}{BW}$	4 = $\sqrt{D_2}$	5 = $\sqrt{\frac{D_2}{BW}}$
6 = $\frac{D_{2N}}{(T_N - T_0)}$ *	7 = $\frac{D_{2N}}{BW\,(T_N - T_0)}$ *	8 = $\frac{\sqrt{D_{2N}}}{(T_N - T_0)}$ *	9 = $\frac{1}{(T_N - T_0)} \sqrt{\frac{D_{2N}}{BW}}$ *
10 = KD_2	11 = $\frac{KD_2}{BW}$	12 = $\sqrt{KD_2}$	13 = $\sqrt{\frac{KD_2}{BW}}$
14 = $\frac{KD_{2N}}{(T_N - T_0)}$ *	15 = $\frac{KD_{2N}}{BW\,(T_N - T_0)}$ *	16 = $\frac{\sqrt{KD_{2N}}}{(T_N - T_0)}$ *	17 = $\frac{1}{(T_N - T_0)} \sqrt{\frac{KD_{2N}}{BW}}$ *
18 = $D_{2n} - D_{2n-1}$ **	19 = $\frac{1}{BW}(D_{2n} - D_{2n-1})$ **	20 = $\sqrt{D_{2n}} - \sqrt{D_{2n-1}}$ **	21 = $\frac{1}{\sqrt{BW}}(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})$ **
22 = $\frac{D_{2n} - D_{2n-1}}{(T_n - T_{n-1})}$ **	23 = $\frac{1}{BW}\frac{(D_{2n} - D_{2n-1})}{(T_n - T_{n-1})}$ **	24 = $\frac{\sqrt{D_{2n}} - \sqrt{D_{2n-1}}}{(T_n - T_{n-1})}$ **	25 = $\frac{1}{\sqrt{BW}}\frac{(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})}{(T_n - T_{n-1})}$ **
26 = $K(D_{2n} - D_{2n-1})$ **	27 = $\frac{K}{BW}(D_{2n} - D_{2n-1})$ **	28 = $\sqrt{K}\,(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})$ **	29 = $\sqrt{\frac{K}{BW}}(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})$ **
30 = $\frac{K(D_{2n} - D_{2n-1})}{(T_n - T_{n-1})}$ **	31 = $\frac{K}{BW}\frac{(D_{2n} - D_{2n-1})}{(T_n - T_{n-1})}$ **	32 = $\frac{\sqrt{K}\,(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})}{(T_n - T_{n-1})}$ **	33 = $\sqrt{\frac{K}{BW}}\frac{(\sqrt{D_{2n}} - \sqrt{D_{2n-1}})}{(T_n - T_0)}$ **

* Data types 6, 7, 8, 9, 14, 15, 16, and 17 will always be last frame only reductions.

** If $\Delta t = (T_n - T_{n-1}) = 1$ second, the following pairs of data types will be numerically equal: 18 and 22, 19 and 23, 20 and 24, 21 and 25, 26 and 30, 27 and 31, 28 and 32, 29 and 33.

Input to the Program

The input to the program will be supplied on IBM tape in a 90 character per record BCD format All records except the group identifying record will be in the following format:

Characters 1 - 7	Time (XXXX XXX) seconds
8 - 9	Code number for type of data (98, 97, or 00)
10	Bank number (1, 2, or 3)
11 14	20 four-place machine count outputs of analyzer
15 - 18	
19 - 22	
⋮	
97 - 90	

Three code numbers are possible in characters 8 and 9: 98 indicates white noise (machine) calibrations; 97 indicates white noise (data) calibration; and 00 indicates data. "Last frame data" is indicated by a minus sign over the bank number in character 10. The 20 four-place outputs of the analyzer comprise one frame of data, i.e., one commutated readout of the twenty filters in one bank. There is a commutated readout for each second of the analysis. (If any output is negative, it is signed in the low order position.)

The first record of a group is the identifying record. The first 8 characters of the identifier are "GROUP" followed by a blank and the two identifying numbers. The remaining (82) characters may contain any information or comment desired, or be blanks.

The 2nd, 3rd, and 4th records of a group are the white noise calibration records for banks 1, 2, and 3 respectively. The 3 white noise (data) calibration records, if present, will be the next records on the tape. Then follow the data records from time T_1 thru time T_N for bank 1, data from time T_1 thru time T_N for bank 2, and data from time T_1 thru time T_N for bank 3. The computer program will be required to sort the dat. records by time and generate identifying frame numbers so that "frame n data" will refer to the output at time T_n of all three banks of filters, numbers 1 thru 60. Groups of data (i.e., successive runs) are stacked on the tapes

Additional information supplied with each group will be:

a. Potentiometer settings used: 3 white noise (machine) pot settings; 3 white noise (data) calibration pot settings if calibrations are used; and 3 data pot settings. All pot settings are two digits, ranging from 0.1 to 9.9.

b. Time duration (T_C) of white noise (data) calibration, ranging from 000.000 to 999.999 seconds.

c. Time duration of data (T_D), ranging from 000.000 to 999.999 seconds.

d. Calibration value (P_N) for white noise (data) calibration, ranging from 000.000 to 999.999, physical units to be specified.

The program request will also indicate whether or not only last frame analysis is desired, as well as the output listing and plotting requirements for each run.

Program Outputs

A. Listings - Two types of listings will be needed. The first will be a continuous list of the data type requested as a function of time, that is, for a given time, t_n, (or frame number), the outputs of all 60 filters at that time (for that frame) will be listed. Such a list may be requested for all time points of the analysis, for an interval of time points, or for a specific time point or frame number. A suggested format to list one time point (frame) is below:

Time t_n	Filter Nos.	Data Type No.
(F7.3 format)	1 - 5	Data $N_{j=1,t_n}$ Data $N_{j=2,t_n}$ ... Data $N_{j=5,t_n}$
	6 - 10	Data $N_{j=6,t_n}$ Data $N_{j=7,t_n}$ ... Data $N_{j=10,t_n}$
	11 - 15	.
	16 - 20	.
	.	.
	.	
	.	
	55 - 60	Data $N_{j=55,t_n}$ Data $N_{j=56,t_n}$ ... Data $N_{j=60,t_n}$

The second type of listing will be a continuous list sorted by filter numbers. The output of a single filter, or filters, or a specified interval of filters may be required.

If plots are requested as the program output, a listing of the plotting information must also be made. This should include all plotting information specified by the program request (time or frame numbers or intervals, data type, etc.) and also all plotting parameters computed by the program (scale factors, board and data offsets, etc.).

B. Plots

There are two types of plots which will be required: bar graphs and continuous line plots. These will be made using the EAI 3440 Dataplotter. Input to the plotter is an IBM tape written at 200BPI in BCD mode. Although the Dataplotter can accept formats of a variety of word and record lengths, the following standard format should be used:

Each record shall contain one data word of 12 BCD characters and shall represent one data point or plotter command function. The first 6 characters are for the X value and the second 6 for Y. The first character is the command X code, the second the sign of X, and characters 3 through 6 are the X data, high order first. Character 7 is the command Y code, character 8 the sign of Y, and characters 9 through 12 are the Y data, high order first. An end of file is required at the beginning as well as the end of each plot. At the end of the last plot on the tape, at least two or three EOF characters should appear.

The following is a list of the plotter command codes available:

COMMAND	X CODE	Y CODE
Skip the data	0	0
Plot the X and Y data	1	1
Set the X and Y scale factors	2	2
Set X scale factor	2	0
Set Y scale factor	0	2
Set X and Y data offset	3	3
Set X data offset	3	0
Set Y data offset	0	3
Set X and Y board offset	4	4
Set X board offset	4	0
Set Y board offset	0	4
Select symbol	0	5
Pen down	0	6
Pen up	0	7
Start new curve	0	8
Stop plot	0	9

Before beginning a plot three sets of variables must be specified to the plotter. These are the X and Y scale factors, the X and Y board offsets, and the X and Y data offsets. The three records specifying this information should be the first three records on the plotter tape. The option should be available either to specify these quantities as load card variables with the program request or to compute them when preparing the plotter tape.

Scale Factors (XXX.X)

The scale factor settings for X and Y determine the number of counts per half-inch of plot. The scale factor used should be determined from the range of the data and the paper size to be used. Normally plots will be requested on either 7 x 10 inch or 10 x 15 inch paper.

Board Offsets (± XX)

The normal origin of the plot board is the center of the board. Other points on the board and their coordinates (in half-inches, X given first) are: Upper left (-30, 30), Upper right (30, 30), Lower right (30, -30), Lower left (-30, -30). The board offset command is used to relocate the origin at any other location on the board. The origin is transferred to the coordinates that are listed with the command. For example, the command to relocate the origin at the lower left hand corner of the board allowing one inch margins on the paper (that is, two half-inches) would be specified by the 12 character record

4 - 00284 - 0028.

The board offset command is usually used to locate the coordinates (0, 0) at the plot origin. The paper will always be placed on the plotting board so that this point is at the board coordinates (0, -20), allowing one inch margins on the paper.

Data Offset (± XXXX)

The data offset command causes the X and Y data offset values to be algebraically added to their respective data point values before the point is plotted. Normally this command is not used. However, it is necessary to specify at the beginning of each plot that the data offset value is (0, 0). That is, the command 3 + 00003 + 0000 should be the third command on each plotting tape

General Instructions

At the beginning of each plotting tape, and between plots on the same tape, there should be one EOF character. After the last plot on the tape there should be at least two or three EOF characters.

The first plotter commands for each plot must be used to set the X and Y scale factors, X and Y board offsets and X and Y data offsets. Even if any of these values are zero, they still must be specifically set to that value.

After these three commands are given, the next command should be "pen up" Then the coordinates of the first data point should be given and repeated about ten times This is to allow the pen time to be positioned in the proper place for the first point before actually beginning the plot Then should follow the command "Pen Down", the first data point repeated once more, and the successive data points in turn

The maximum efficiency on the plotter is achieved by plotting at high speed. However, in order to plot accurately at high speeds, the points to be plotted must be close together Accurate high speed plots can best be produced when the vector distance between consecutive points on the line is no greater than 1/8 inch Thus any line segment (particularly the sides of bars on bar graphs) should be specified by a succession of points no more than 1/8 inch apart

Continuous Line Plots

Line plots may be requested of the continuous output data of any filter as a function of time (or frame number) Time may run from t_0 thru t_N (time of last frame) or over any specified time interval The data types which may be plotted in this way include types 2 thru 5, 10 thru 13, and any of the incremental data types 18 thru 33 (Data types 6 thru 9 and 14 thru 17 are "last frame only" data types, and are always plotted as bar graphs of data vs. filter number)

The X scale factor for line plots is determined from the time interval to be plotted and the paper size used The Y scale factor is determined from the maximum value the data reaches For data types 2 thru 5 and 10 thru 13, this maximum is reached in the last frame of the analysis; for any of the incremental data types 18 thru 33, the program must search for the maximum value. For relative data (uncalibrated), that is, any of types 2 thru 5 and 18 thru 25, the data maximum should be set equal to 100% of full scale, and data values plotted as percentage points For calibrated data there should be the option available to insert a desired maximum value as 100% full scale, or the factor should be chosen knowing the data maximum and paper size to be used so that the divisions on the graph paper correspond to convenient (even) data values.

Bar Graphs

The outputs of data types 6 thru 9 and 14 thru 17 will always be plotted as bar graphs In addition, a bar graph of any single frame or frames of any data type vs filter numbers may be requested. The filter interval to be graphed may include all 60 filters, or any bank of twenty at a time.

Plotting a bar graph is more complicated than plotting a continuous line plot. The bars will be drawn with the plotter in "line plot" mode Thus each line should be specified by a succession of points, no more than 1/8 inch apart, to maintain high speed and accuracy on the plotter.

A convenient shape of bar graph is achieved by making the bars twice as wide as the spaces between them This leads to the formula;

$(N[S + B] - S) =$ full width of graph,

where N is the number of bars (20 or 60)

S is the width of a space

$B = 2S$ is the width of a bar.

The formula should be solved for S, the width of a space between bars, subject to the restriction that $S \leq 1/4$ inch, to maintain a balanced appearance.

SYMBOLS USED IN COMPUTATIONAL PROCEDURE

B = 1, 2, 3 = bank number

J = 1, 2, ··· , 60 = filter number

PSM(B), PSC(B), PSD(B) = Pot Settings for Machine correction, Calibration, and Data for bank (B), respectively (load card variables)

PFM(B), PFC(B), PFD(B) = Pot Factors for Machine correction, Calibration, and Data for bank (B), respectively

W'(J) - Output, in machine counts, of last frame White Noise (machine) calibration of the J^{th} filter

R(J) - White Noise correction factor for J^{th} filter

BW(J) - Bandwidth of J^{th} filter

K - Calibration constant computed if data calibration is present

P_W - Total power output of white noise generator, used in computing constant K (load card variable)

T_C - Time duration (seconds) of Calibration, used in computing K (load card variable)

D(J) - Output, in machine counts, of last frame white noise Data calibration of the J^{th} filter It should be corrected by its appropriate pot factors and white noise (machine) correction factors before computing K.

COMPUTATIONAL PROCEDURE

Note: Wherever index B occurs, do B = 1, 2, 3.

Wherever index J occurs, do J = 1, 2, ···, 60.

If both J and B occur in a product, B will be determined from the following relationships:

for $1 \leq J \leq 20$, B = 1

for $21 \leq J \leq 40$, B = 2

for $41 \leq J \leq 60$, B = 3.

1. Compute pot factors

$$PF = \left(\frac{1}{PS}\right)^2$$

2. Compute white noise (machine) correction factors, R(J):

Let M(J) = [WN(J)] x [PFW(B)]

Let R(1) = 1.

$$R(J) = \left[10^{(.05)(J-1)}\right]\left(\frac{M(1)}{M(J)}\right)$$

for $2 \leq J \leq 60$

3. Compute combined correction factor for each filter, F(J):

F(J) = PFW(B) x R(J)

4. Compute filter bandwidths, BW(J):

$$BW(J) = (1.22)\left[10^{(J-1)(.05)}\right]$$

5. If white noise (data) calibration is present, compute constant K

a. Correct data calibration D(J) from J^{th} filter

D(J) = [Last frame calibration value machine counts of J^{th} filter] x PFC(B) x R(J)

b. $K(J) = (2.35849 \times 10^{-4}) \dfrac{P_N \, T_C \, BW(J)}{D(J)}$

for $(J) = 1, 2, \cdots, 60$

c. $K = \dfrac{1}{60} \sum_{J=1}^{60} K(J)$

6. For each frame of data and each filter compute the requested data type. Data n(j,t) where j is the filter number and t is the (time/frame number) index.

7. Prepare requested output listings and plotter tapes.